$\boxed{\text{FEM for Parabolic problems.}}$

$$u_t = \nabla(\beta\nabla u) + f(x,y,t) \qquad (x,y) \in \Omega.$$

$$u(x,y,0) = u_0(x,y) = \underset{\frown}{0} \qquad IC$$

$$u(x,y,t)\big|_\Gamma = u_1(\Gamma, t) \qquad BC.$$

Weak form? Galerkin form.

$$\iint_\Omega u_t \, v \, dxdy = \iint v\nabla\beta(\nabla u)\,dxdy + \iint f(x,y,t)\,v\,dxdy$$

$$= \underbrace{\int_\Gamma \beta \frac{\partial u}{\partial n} v\,ds}_{=0} - \iint_\Omega \beta\,\nabla u\cdot\nabla v\,dxdy + \iint_\Omega f(x,y)$$

or:

$$(u_t, v) = -a(u,v) + (f,v). \qquad u, v \in H_0^1(\Omega)$$

Choose finite element space

$$V_h \subset V_0^1(\Omega) \qquad \text{e.g. Linear basis function.}$$

$$\varphi_1, \varphi_2, \cdots, \varphi_M.$$

$$u_h(x,y,t) = \sum_{j=1}^{M} \xi_j(t)\,\varphi_j(x,y).$$

plut into the Galerkin form:

$$\left(\sum_{j=1}^{M} \xi_j'(t)\,\varphi_j(x,y),\ \varphi_i(x,y)\right) + \left(\sum_{j=1}^{M} \xi_j(t)\,\varphi_j(x,y),\ \varphi_i(x,y)\right) = (f, \varphi_i(x,y))$$

Linear system of Equations.

$$\underline{\begin{bmatrix} (\varphi_1,\varphi_1) & (\varphi_1,\varphi_2) & \cdots & (\varphi_1,\varphi_M) \\ (\varphi_2,\varphi_1) & (\varphi_2,\varphi_2) & \cdots & (\varphi_2,\varphi_M) \\ \vdots & & \cdots & \\ (\varphi_M,\varphi_1) & (\varphi_M,\varphi_2) & \cdots & (\varphi_M,\varphi_M) \end{bmatrix}} \begin{bmatrix} \xi_1'(t) \\ \xi_2'(t) \\ \vdots \\ \xi_M'(t) \end{bmatrix} + \underline{\begin{bmatrix} a(\varphi_1,\varphi_1) & a(\varphi_1,\varphi_2) & \cdots & a(\varphi_1,\varphi_M) \\ a(\varphi_2,\varphi_1) & a(\varphi_2,\varphi_2) & \cdots & a(\varphi_2,\varphi_M) \\ & \cdots & & \\ a(\varphi_M,\varphi_1) & a(\varphi_M,\varphi_2) & \cdots & a(\varphi_M,\varphi_M) \end{bmatrix}}$$

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_M(t) \end{bmatrix} = \begin{bmatrix} (f,\varphi_1) \\ (f,\varphi_2) \\ \vdots \\ (f,\varphi_M) \end{bmatrix}$$

$$u_h(x,y,t) = \sum_{j=1}^{M} \xi_j(t)\,\varphi_j(x,y)$$

nodal point

$$\xi(0) = u(N_i, 0) \qquad B\dot{\xi}(t) + A\xi = F(t)$$

Both A and B are S.P.D. B is symmetric

dsode.f

System of ODE of first kind.

Numerical Methods

A: Numerical ODEs

Can prove: The ~~syst~~ ODE system is very stiff.

$$\dot{\xi}(t) = -\underline{B^{-1}A}\,\xi + B^{-1}F(t)$$

The eigenvalues of $B^{-1}A$ have very different magnitude.

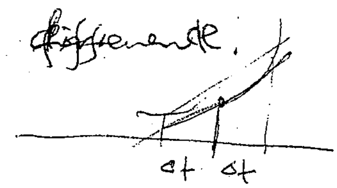
B. Implicit method. $\xi(0) = \xi^0$

$$t^0 = 0, \quad t^1 = t^0 + \Delta t_0, \quad t^2 = t^1 + \Delta t_1, \quad \ldots, \quad t^{n+1} = t^n + \Delta t_n.$$

$$\xi^0 = u(N_i, 0) \qquad \xi^1 = ? \qquad \cdots \qquad \xi^{n+1}$$

Time marching method $\quad t^n \to t^{n+1}, \quad \xi^n \to \xi^{n+1}$.

Implicit Euler's method: Backward ~~difference~~.



$$f'(t) \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} \qquad \text{Forward}$$

$$f'(t) \approx \frac{f(t) - f(t-\Delta t)}{\Delta t} \qquad \text{Backward.}$$

Not recommended: $B\dfrac{\xi^{n+1}-\xi^n}{\Delta t^n} + A\xi^n = F(t^n)$ Explicit $\qquad \dot{\xi}(t) = -BA^{-1}\xi + B^{-1}F$

✓ $$B\,\frac{\xi^{n+1} - \xi^n}{\Delta t^n} + A\xi^{n+1} = F(t^n)$$

Implicit Euler.

or $$(B + \Delta t^n A)\,\xi^{n+1} = B\xi^n + F(t^n)$$

Stabilility Unconditionally stable.

Explicit. would require $\Delta t^n \le c h^2$ very small.

Accuracy. $O(\Delta t, h^2)$ for linear basis function

Crank-Nicholson scheme. Average between the time t^n t^{n+1}.

$$B\dot{\xi}(t) + A\xi = F$$

$$B\,\frac{\xi^{n+1}-\xi^n}{\Delta t^n} + A\,\frac{1}{2}(\xi^n + \xi^{n+1}) = \frac{1}{2}\left[F(t^n + t^{n+1})\right]$$

or $$\left(B + \frac{1}{2}\Delta t^n A\right)\xi^{n+1} = \left(B - \frac{1}{2}\Delta t^n A\right)\xi^n + \frac{1}{2}\left[F(t^n) + F(t^{n+1})\right]$$

Unconditionally stable, Accuracy $O(\Delta t^2, h^2)$.

Second order accurate both in space and time.

A clever transform: B is S.P.D. $B^{\frac{1}{2}}$ exists.

$$B\dot{\xi}(t) + A\xi = F$$

$$B^{\frac{1}{2}} B^{\frac{1}{2}}\dot{\xi}(t) + A\xi = F$$

$$B^{\frac{1}{2}}\dot{\xi}(t) + B^{-\frac{1}{2}} A B^{-\frac{1}{2}} B^{\frac{1}{2}}\xi(t) = B^{-\frac{1}{2}} F$$

Let $\eta(t) = B^{\frac{1}{2}}\xi(t)$, we get

$$\dot{\eta}(t) = -B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\eta(t) + B^{-\frac{1}{2}} F$$
$$= -\hat{A}\,\eta(t) + \tilde{F}.$$ $\hat{A}$ is a S.P.D.

Stokes Equations

$$\begin{cases} \nabla p = \mu \Delta \vec{u} + \vec{F} & (x,y) \in \Omega \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}\big|_{\partial\Omega} = 0 \end{cases}$$

Renolds number is very small, inertia term is negnected, very viscous, small velocity, creeping flow.

Finite element method: A eliminate p.

$\vec{u} = (u_1, u_2)$

$$\iint_\Omega \nabla p \cdot \vec{v}\, dx\,dy = \iint_\Omega \mu \Delta \vec{u} \cdot \vec{v}\, dx\,dy + \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$\begin{cases} \nabla \cdot \vec{v} = 0 \\ \vec{v}\big|_{\partial\Omega} = 0 \end{cases} \quad \text{tough constraints.}$$

$$\Rightarrow \int_{\partial\Omega} (\nabla p \cdot \vec{n})\, \vec{v}\, dx\,dy - \iint_\Omega p\, \nabla \cdot \vec{v}\, dx\,dy$$

$$= -\iint_\Omega \mu (\nabla \vec{u} \cdot \nabla \vec{v})\, dx\,dy + \int_{\partial\Omega} \mu \frac{\partial \vec{u}}{\partial n} \vec{v} + \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$\Rightarrow \mu \iint_\Omega \left\{ (\nabla u_1 \cdot \nabla v_1) + (\nabla u_2 \cdot \nabla v_2) \right\} dx\,dy = \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$V = \left\{ \vec{v} \in [H_0^1(\Omega)]^2,\ \nabla \cdot \vec{u} = 0,\ \vec{v}\big|_{\partial\Omega} = 0 \right\}$$

Difficulty: How to find V space,

Use the stream function

$$\vec{v} = \operatorname{rot} \varphi = \left\langle \frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x} \right\rangle$$

then $\nabla\cdot\vec{v} = \mathrm{div}\,\vec{v} = \frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x} = 0$

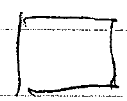
Let W_h be a finite dimensional subspace of $H_0^2(\Omega) = \{\vec{v}(x),\ \vec{v}(x)|_{\partial\Omega}=0,\ \vec{v}(x)\in H^2(\Omega)\}$

Then $V_h = \left\{\vec{v},\ \vec{v} = \begin{bmatrix}\frac{\partial\varphi}{\partial y}\\ -\frac{\partial\varphi}{\partial x}\end{bmatrix},\ \vec{v}|_{\partial\Omega}=0\right\}$

Error estimate

$$\|u-u_h\|_{H^1(\Omega)} \le C h^4 \|u\|_{H^5(\Omega)}$$

Note that. If the boundary conditions are periodic for all variables,



$$\vec{u}(a,y) = \vec{u}(b,y),\quad \vec{u}(x,c) = \vec{u}(x,d)$$
$$p(a,y) = p(b,y),\quad p(x,c) = p(x,d)$$

then we have

$$\Delta p = \nabla\cdot\mu\overset{=0}{\Delta u} + \nabla\cdot\vec{F}$$

$$\Rightarrow \begin{cases}\Delta p = \nabla\cdot\vec{F}\\ \mu\Delta u_1 = p_x + F_x\\ \mu\Delta u_2 = p_y + F_y\end{cases}$$

Method Mixed finite element method

Do not impose the incompressibility condition for $\vec{V}$.

$$\iint \nabla p \cdot \vec{v}\, dx\,dy = \iint \mu \Delta \vec{u} \cdot \vec{v} + \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$+ \int_{\partial\Omega} p\, \vec{n} \cdot \vec{v}\, ds - \iint p\, \nabla \cdot \vec{v}\, dx\,dy$$

$$= \int_{\partial\Omega} \mu\, \nabla \vec{u} \cdot v - \iint_{\Omega} \nabla \vec{u} \cdot \nabla \vec{v}\, dx\,dy + \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$\Rightarrow \iint_{\Omega} \nabla \vec{u} \cdot \nabla \vec{v}\, dx\,dy - \iint_{\Omega} p\, \nabla \cdot \vec{v}\, dx\,dy = \iint_{\Omega} \vec{F} \cdot \vec{v}\, dx\,dy$$

or $(\nabla \vec{u}, \nabla \vec{v}) - (p, \operatorname{div} \vec{v}) = (\vec{F}, \vec{v})$

$$(\nabla w, \nabla v) = \sum_{i=1}^{2} \int_{\Omega} (\nabla w_i \cdot \nabla v_i)\, dx\,dy$$

Second equation

$$\nabla \cdot \vec{u} = 0, \qquad \operatorname{div} \vec{u} = 0$$

$$\iint (q \cdot \operatorname{div} \vec{u})\, dx\,dy = 0$$

Impose additional equation for the pressure

$$\iint_{\Omega} p\, dx\,dy = 0$$

or $p(x^*, y^*) = p_0$ fixe at the point

$$V = \{ \vec{v} \in [H^1(\Omega)]^2, \ \vec{v}|_{\partial\Omega} = 0 \}$$

$$H = \{ q \in L_2(\Omega), \ \iint_{\Omega} q\, dx\,dy = 0 \}$$

Different sub-spaces, so it is called a mixed finite elements.

Stability requirement.

$$\|u_h\|_{H^1(\Omega)} + \|p_h\|_{L^2(\Omega)} \le C \underbrace{\sup_{v \in H^1(\Omega)} \frac{(f,v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}}}_{\|f\|_{H^{-1}(\Omega)}}$$

Then we have

$$\|u_h\|^2_{H^1(\Omega)} \le (f, u_h) \le \|f\|_{H^{-1}(\Omega)} \|u_h\|_{H^1(\Omega)}$$

or $\quad \|u_h\|_{H^1(\Omega)} \le \|f\|_{H^{-1}(\Omega)}$

Numerics: $\quad V_h = \{ \vec{v} \in V : \vec{v}|_K \in [Q_2(K)]^2, \ \forall K \in T_h \}$

$\quad H_h = \{ q \in H : q|_K \in Q_0(K), \ \forall K \in T_h \}$.

or $\quad V_h = \{ \vec{v} \in V, \ \vec{v}|_K \in [Q_1(K)]^2, \ \forall K \in T_h \}$

$\quad H_h = \{ q \in H, \ q|_K \in Q_0(K), \ \forall K \in T_h \}$.

or

$\quad V_h \in \{ \vec{v} \in V; \ \vec{v}|_K \in [Q_2(K)]^2; \ \forall K \in T_h \}$

$\quad H_h \in \{ q \in H, \ q|_K \in P_1(K), \ \forall K \in T_h \}$.

Reference: Numerical solution of PDE by the finite element method

Claes Johnson.

Cambridge University Press, 1994