SELECTED SOLUTION TO HW #2 AND MIDTERM I

- Problem 1.

$$
\begin{aligned}
u^{(4)}(x) &\simeq \frac{1}{h^4}\Big(u(x-2h)-4u(x-h)+6u(x)-4u(x+h)+u(x+2h)\Big)\\
u^{(3)}(x) &\simeq \frac{1}{h^3}\Big(-6u(x)+3u(x+h)-3u(x+2h)+u(x+3h)\Big)
\end{aligned}
$$

- Problem 2. We can set $u(x_1)$, $u(x_1)$, $\cdots$, $u(x_n)$ as unknowns. Once we know $u(x_n)$, then $u(a) = u(x_0) = u(x_n)$. The first finite difference equation is

$$\frac{U_0 - 2U_1 + U_2}{h^2} = f(x_1), \quad \text{or} \quad \frac{-2U_1 + U_2 - U_n}{h^2} = f(x_1)$$

since $u(a) = u(b)$ or $U_0 = U_n$. Similarly, the last equation is

$$\frac{U_{n-1} - 2U_n + U_{n+1}}{h^2} = f(x_n), \quad \text{or} \quad \frac{U_1 - 2U_{n-1} + U_n}{h^2} = f(x_n)$$

since $u(a+h) = u(b+h)$ or $U_{n+1} = U_1$.

Thus the coefficient matrix is:

$$A = \frac{1}{h^2}\begin{pmatrix} -2-h^2q_1 & 1 & 0 & \cdots & 1 \\ 1 & -2-h^2q_2 & 1 & \cdots & \cdots \\ 0 & 1 & -2-h^2q_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & -2-h^2q_n \end{pmatrix}$$

When $q(x) = 0$, the solution may not exist. Even if the solution $u(x)$ exists, it is not unique since $u(x) + C$ will be also a solution. The compatibility condition is the condition such that the solution exists. In this case, is is $\int_a^b f(x)dx = 0$. When the solution does exist, we can specify the solution at one point to make it unique.

- Problem 3. The coefficient matrix has 6 rows and 6 columns since there are 6 unknowns.