MA 584
Homework #2
Due Tuesday, October 6, 2009

1. Given a set of points (x_i, u_i), $i = 0, 1, \cdots N$, we can find the Lagrange interpolation polynomial from the formula

$$p(x) = \sum_{i=0}^{N} l_i(x)\, u_i, \qquad l_i(x) = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}. \tag{1}$$

By differentiating $p(x)$ with respect to x, we can get different finite difference formulas for approximating different derivatives. Assume that we have a uniform mesh, i.e. $x_{i+1} - x_i = x_i - x_{i-1} = \cdots = x_1 - x_0 = h$. Derive a central finite difference formula for $u^{(4)}$ and a one-sided finite difference formula for $u^{(3)}$.

2. Derive the finite difference method for

$$\begin{aligned} u''(x) - q(x)u(x) &= f(x), \quad a < x < b, && (2) \\ u(a) &= u(b), \quad \text{periodic BC}, && (3) \end{aligned}$$

using the central difference scheme and a uniform grid. Write down the system of equations $A_h U = F$. How many unknowns are there without redundant? Is the coefficient matrix A_h tri-diagonal? **Hint:** Note that $U_0 = U_n$. Set unknowns as U_1, U_2, $\cdots$, U_n. If $q(x) = 0$, does the solution exist? Derive a compatibility condition for which the solution exists. If the solution exists, is it unique? How do we modify the finite difference method to make the solution unique?

3. Write down the coefficient matrix of the finite difference method using the standard central five point stencil with the Red-Black ***or*** the Natural row ordering for the Poisson equation defined on the rectangle $[a,\ b] \times [c,\ d]$. Take $m = n = 3$ and assume a Dirichlet boundary condition at $x = a$, $y = c$ and $y = d$, and a Neumann boundary condition $\frac{\partial u}{\partial n} = g(y)$ at $x = b$. Use the ghost point method to deal with the Neumann boundary condition.

4. (Programming Part) Implement and compare the Gauss-Seidel, and the SOR (trying to find the best ω by testing), methods for the following elliptic problem:

$$u_{xx} + p(x,y)u_{yy} + r(x,y)u(x,y) = f(x,y)$$
$$a < x < b; \qquad c < y < d,$$

with the following boundary conditions:

$$u(a,y) = 0, \quad u(x,c) = 0, \quad u(x,d) = 0, \quad \frac{\partial u}{\partial x}(b,y) = -\pi \sin(\pi y).$$

Test and debug your code for the case $\quad 0 \le x, y \le 1$, and

$$p(x,y) = (1 + x^2 + y^2), \qquad r(x,y) = -xy.$$

The the source term $f(x,y)$ is determined from the exact solution

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Do the grid refinement analysis for $n = 16$, $n = 32$, and $n = 64$ (if possible) in the infinity norm (**Hint:** In matlab, use $max(max(abs(e)))$). Take the tolerance as 10^{-5}. Does the method behave like a second order method? Compare also the number of iterations and test the optimal relaxation factor ω. Plot the solution and the error for $n = 32$.

Having made sure that you code is working correctly, try your code with a point source $f(x,y) = \delta(x-0.5)\delta(y-0.5)$ and $u_x = -1$ at $x = 1$, with $p(x,y) = 1$ and $r(x,y) = 0$. Note that the $u(x,y)$ can be interpreted as the steady state temperature distribution of a room with insulated wall on three sides, a constant heat flow in from one side, and a point source (a heater for example) in the room. Note that the heat source can be expressed as $f(n/2, n/2) = 1/h^2$, and $f(i,j) = 0$ for other grid points. Use the mesh and contour plots to visualize the solution for $n = 36$ $(mesh(x,y,u), contour(x,y,u,30))$.

Hint: Please try to work out the code(s) by yourself. However, if you really do not know how to start, you can send me an e-mail asking for a sample code. **Note:** If you can use a multigrid solver for this problem, then you do not need to use Gauss-Seidel, and the SOR method, see the extra credit problems.

Extra Credit: Choose one from the following:

5. Search the Internet to find a fast Poisson solver or multigrid method and test it.