MA580 Midterm Review

Concepts:

Round-off errors; floating point arithmetic; absolute and relative errors; Hornet's algorithm for evaluating polynomial; catastrophic cancellations and how to avoid them; Augmented matrix; Pivot element; Pivot row; Partial pivoting; Unit lower/upper triangular matrix; Lower/upper triangular matrix; Backward/Forward substitution; Elementary permutation matrix P_{ij}, (P_{ij}^T =?, P_{ij}^{-1} =?, P_{ij}^2 =?, $det(P_{ij})$ =?, $P_{ij}A$ =?, AP_{ij} =?); Permutation matrix P, (P^{-1} =?, $det(P)$ =?); Strictly row diagonally dominant (SRDD); Symmetric positive definite (S.P.D, how to tell?); Principal/Leading sub-matrices; Choleski decomposition (LL^T); Band matrix; Tridiagonal matrix; Backward error analysis; Perturbation analysis

Vector norms; The 1, 2 (Euclidean), and ∞ vector norms; Matrix norms; 1, 2, ∞ matrix norms; Subordinate (associate, natural) matrix norm; Vector and Matrix convergence; Condition Number $cond(A)$; Well/Ill-conditioned matrix; Residual Vectori, Direct method; Iterative method; Vector and Matrix convergence; Iteration matrix; Spectral radius $\rho(A)$; Condition Number $cond(A)$; Well/Ill-conditioned matrix; Residual Vector.

Direct Methods:

Method	A	**Pivoting**	**Storage**	**Can fail?**	**Operations**	$det(A)$
GE to $[A:b]$	general	Yes	$O(n^2)$	No	$O(\frac{n^3}{3})$	$(-1)^s \prod_{i=1}^{n} a_{ii}$
$P^T LU$	general	Yes	$O(n^2)$	No	$O(\frac{n^3}{3})$	$(-1)^s \prod_{i=1}^{n} a_{ii}$
Direct LU	general	No	$O(n^2)$	Yes	$O(\frac{n^3}{3})$	$\prod_{i=1}^{n} u_{ii}$
	SRDD	No	$O(n^2)$	No	$O(\frac{n^3}{3})$	$\prod_{i=1}^{n} u_{ii}$
Crout	Tridiagonal	No	$O(3n)$	Yes	$O(2n)$	$\prod_{i=1}^{n} u_{ii}$
LL^T	S.P.D	No	$O(\frac{n^2}{2})$	No	$O(\frac{n^3}{6})$	$\prod_{i=1}^{n} l_{ii}^2$
Back-Substi.	Triangular	–	$O(\frac{n^2}{2})$	–	$O(\frac{n^2}{2})$	–

Pivoting strategy:

Partial/maximal-column pivoting:

$$|a_{pk}| = \max_{k \le i \le n} |a_{ik}|.$$

From $PA = LU$ factorization to solve $Ax = b$:

- Compute Pb,
- Forward substitution: $Ly = Pb$,
- Backward substitution: $Ux = y$.

Stationary Iterative Algorithms: $x^{(k)} = T\,x^{(k-1)} + c, \quad k = 1, 2, \cdots$

Jacobi Iterative Method: $T_g = D^{-1}(L+U); \quad c = D^{-1}b.$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \quad i = 1, 2, \cdots, n.$$

The Jacobi method converges if A is strictly row diagonally dominant.

Gauss-Seidel Iterative Method: $T_g = (D-L)^{-1}U; \quad c = (D-L)^{-1}b.$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \quad i = 1, 2, \cdots, n.$$

The Jacobi method converges if A is strictly row diagonally dominant or symmetric positive definite.

SOR Iterative Method: $T_\omega = (D-\omega L)^{-1}[(1-\omega)D + \omega U]; \quad c = \omega(D-\omega L)^{-1}b.$

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \quad i = 1, 2, \cdots, n.$$

The SOR method converges if A is symmetric positive definite and $0 < \omega < 2$.

Theories:

- HW problems.
- Questions imposed in class.
- How to avid loss of accuracy of floating number operations. For example,

$$\sqrt{a+\delta} - \sqrt{a}, \qquad f(x+h) - f(x).$$

- Find an upper bound for $\|A - fl(A)\|$ and its effect on the solution of $Ax = b$.
- Show that $cond(A) \geq 1$.
- Show that $\|Ax\| \leq \|A\|\|x\|$, $\|AB\| \leq \|A\|\|B\|$.

- Show that $cond(A) \geq 1$.
- Error estimates of the direct methods and iterative methods.
- Orthogonal matrices ($Q^{-1} = Q^T$, $Q^T Q = I$, $||Qx||_2 = ||x||_2$, $||QA||_2 = ||A||_2$, $\kappa_2(QA) = \kappa_2(A)$ etc.)
- Show that $\rho(R) \leq \|R\|$.
- If $\|R\| < 1$, show the convergence of the iterative method.

Supplementary exercises

1. Perform Gauss elimination with partial column pivoting on the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 2 & 4 & 2 \end{bmatrix}.$$

   (a) What is the LU factorization of PA (where $PA = LU$)?
   (b) Compute the direct factorization $A = LU$.
   (c) Approximately how many operations are required in (a) and (b).
   (d) Compute the determinant of A from (a) and (b).
   (e) Use your factorization results from (a) and (b) to solve $Ax = b$, where $b^T = \begin{bmatrix} 2 & -4 & 6 \end{bmatrix}$.
   (f) Find $\|A\|_p$, $cond_p(A)$ for $p = 1, 2, \infty$.

2. List at least three kinds of matrices for which pivoting strategies may not be necessary and explain why.

3. Suppose

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1/5 & 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Can L_1 or L_3 be a Gauss transformation matrix with partial pivoting? Why?
   (b) Compute L_1^{-1}, L_3^{-1}, $L_1 L_3$, and $L_1^{-1} L_3^{-1}$.
   (c) Compute P^{-1}, P^T, P^2, PL_3, and PL_3P.

4. For the following matrices

$$A = \begin{bmatrix} 3 & -1 & \alpha \\ -1 & \beta & 1/2 \\ 1 & 1/2 & \gamma \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 1 & \alpha & -1 & 1 \\ 0 & -1 & \beta & \gamma \\ 1 & 1 & 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \gamma & -2 \\ \beta & 2 & \gamma \\ -2 & 5 & 4 \end{bmatrix},$$

can you choose the parameters so that the matrix is

(a) symmetric;

(b) strictly row diagonally dominant;

(c) symmetric positive definite?

5. Suppose $A = LDL^T$, where L is a unit lower triangular matrix.

(a) Is A symmetric?

(b) When is A a symmetric positive definite matrix?

(c) What are the orders of operations (multiplication/division, addition/subtraction) needed for such decomposition?

(d) Can you get $A = LL^T$ factorization from $A = LDL^T$ if A is a S.P.D? How?

6. Derive $A = LU$ decomposition, where U **is a unit upper triangular matrix**. That is to derive the recursive relation for

$$\begin{aligned} &l_{ij}, \qquad && i = j,\, j+1, \cdots, n, \\ &u_{ij}, \qquad && j = i+1, \cdots, n, \end{aligned}$$

(a) Write a pseudo code for your algorithm.

(b) How many operations (multiplications/divisions and addition/subtractions) are required in your algorithm.

(c) Outline how to use such a decomposition to solve $Ax = b$ and compute the determinant of A.

7. For the following model matrices, what kind of matrix-factorization would you like to use for solving the linear system of equations? Analyze your choices (operation count, storage, pivoting etc).

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0.01 & 3 & 0 & -4 \\ 1 & 2 & 1 & 2 \\ -1 & 0 & 3 & -2 \\ 5 & -2 & 3 & 6 \end{bmatrix}.$$

8. Let $A^{(2)}$ is the matrix obtained after one step Gauss elimination applied to a matrix A, that is

$$a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}.$$

   (a) Show that

$$\max_{ij} |a_{ij}^{(2)}| \leq 2 \max_{ij} |a_{ij}| \qquad (1)$$

   if partial pivoting is used.

   (b) Show that without pivoting, (1) is still true if A is row diagonally dominant.

9. Write a pseudo-code to solve the quadratic equation $ax^2 + bx + c = 0$. You should consider all the possible scenarios and avoid possible large round-off errors.