Some Factors of Eigenvalues and the Convergence of Stationary Iterative Methods

Preliminary:

- If $A \in R^{n \times n}$, if there are $\lambda \in R$ and $x \neq 0 \in R^{n \times 1}$ such that $Ax = \lambda x$, then λ is called an eigenvalue of A, x is one eigenvector corresponding to λ.
- There are n eigenvalues of $A \in R^{n \times n}$. Denote as $\lambda_1(A)$, $\lambda_2(A)$, $\cdots$, $\lambda_n(A)$. They are the roots of the polynomial $det(\lambda I - A) = 0$.
- If

  $$A = \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix}$$

  Then the eigenvalues of A are a_{11} plus the eigenvalues of A_1.
- The eigenvalues are diagonal elements for diagonal, upper/lower, triangular matrices.
- The eigenvalues of A^m are $\lambda_i^m(A)$.
- If A is symmetric, then all eigenvalues are real.
- If A is symmetric, then $\|A\|_2 = \max_i |\lambda_i(A)|, \quad \|A^{-1}\|_2 = \frac{1}{\min_i |\lambda_i(A)|}$.
- $\rho(A) = \max_i |\lambda_i(A)|$ is called the spectral radius of A.

Stationary Iterative Method: $x^{k+1} = Rx^k + c$.

- Consistency condition: $I - R$ is nonsingular.
- Necessary and sufficient condition of the convergence: $\rho(R) < 1$.
- Sufficient condition of the convergence: there is a subordinate matrix norm such that $\|R\| < 1$.
- Sufficient conditions of convergence for special iterative method.
  - If A is strictly row diagonally dominant, then both Jacobi and Gauss-Seidel methods converge. Gauss-Seidel iteration converges faster than Jacobi
  - If A is weakly row diagonally dominant and irreducible, then both Jacobi and Gauss-Seidel methods converge.
  - If A is symmetric positive definite, then Gauss-Seidel method converges.
  - Usually, from the eigenvalues of A, we can get an upper bound for the spectral radius, and we can prove the convergence properties for Jacobi, Gauss-Seidel, and SOR(ω) methods.