MA/CSC 580, Homework #5
Due Tuesday, November 25, 2008

1. Assume that $\|R\| < 1$, show that the iterative method $\mathbf{x}^{k+1} = R\mathbf{x}^k + \mathbf{c}$ has the error estimate

$$\|\mathbf{x}^k - \mathbf{x}^*\| \le \frac{\|R\|^k}{1 - \|R\|} \|\mathbf{x}^1 - \mathbf{x}^0\|,$$

where $\mathbf{x}^*$ is the solution to $\mathbf{x} = R\mathbf{x} + \mathbf{c}$. Note that in this estimate, the true solution is not involved in the right hand side. **Hint:** First show that $\mathbf{x}^{k+1} - \mathbf{x}^k = R(\mathbf{x}^k - \mathbf{x}^{k-1}) = R^k(\mathbf{x}^1 - \mathbf{x}^0)$, and then $R\left(\mathbf{x}^k - \mathbf{x}^*\right) - \left(\mathbf{x}^k - \mathbf{x}^*\right) = R^k(\mathbf{x}^1 - \mathbf{x}^0)$.

2.

(a) Use the Gershgorin's theorem to locate the intervals that contain the eigenvalues of A

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -6 & 0 \\ 0 & 1 & 9 \end{bmatrix}.$$

(b) Can A have complex eigenvalues? Why?

(c) Is A diagonalizable? Why?

(d) Apply one step Power method using the 2-norm, and the x_p notation.

(e) Assume that eigenvalues A satisfy $|\lambda_1| > |\lambda_2| > |\lambda_3|$, apply one step ***shifted inverse Power method*** to approximate the eigenvalue λ_2 and its eigenvector with initial guess $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

Hint: You can use Matlba to find $(A + 6I)^{-1}$.

3. Write a computer program to find the least dominant eigenvalue of the matrix A and corresponding the unit eigenvector. Test your code for the following two matrices:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix}$$

Tabulate the number of iterations, the relative error of the eigenvalue, and the residue vector $\|Ax - \lambda x\|_2$ for $n = 10, 20, 40, 80, 160$. For the second matrix, also try $n = 11, 21, 41, 81, 161$. Explain your results. **Hint:** Use the Lemma learned in class to find the exact eigenvalues of A for the first matrix; and the Matlab function $eig(A)$ for the second matrix.

4. Let $x = \begin{bmatrix} 3 & 0 & -1 & 2 \end{bmatrix}^T$, and $y = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T$.

(a) Is there a Householder matrix P such that $Px = y$? Explain.

(b) Let $\tilde{y} = \alpha y$, find the scalar α and a Householder matrix P such that $Px = \tilde{y}$.

5.

(a) Show that the least squares solution of

$$\begin{bmatrix} 0_1 \\ B \\ 0_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is $x^* = B^{-1}b_2$, where 0_1 and 0_2 are zero matrices, and B is an invertible square matrix.

(b) Use the result you proved in Part a to solve the following least squares problems and calculate the 2-norm of the residual vector for each solution you computed.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} ; \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

6. Use the QR method to solve the over-determined system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix} .$$

(a) $b = \begin{bmatrix} 3 & 0 & 0 & 6 & -8 \end{bmatrix}^T$.

(b) $b = e_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$.

Calculate the 2-norm of the residual vector for each solution you computed.

7. Find the singular value decomposition for the matrix (with the process)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} .$$