Selected solutions

1 The coefficient matrix of the red-black ordering is

$$A = \begin{pmatrix} D_1 & B \\ B^T & D_2 \end{pmatrix} \qquad D_i = -\frac{4}{h^2} I, \quad i = 1, 2$$

B is a sparse matrix in which each row has at most 4 non-zero entries. The size of the matrix is $(n-1)^2$ by $(n-1)^2$ with non-zero entries of $O(5n^2)$. The $SOR(\omega)$ at k-th iteration can be written as

$$u_{ij}^{k+1} = u_{ij}^k, \quad i, j = 1, 2, \cdots, n-1.$$

$$u_{ij}^{k+1} = (1-\omega)u_{ij}^k + \frac{\omega}{4}\left(u_{i-1,j}^{k+1} + u_{i+1,j}^{k+1} + u_{i,j-1}^{k+1} + u_{i,j+1}^{k+1} - h^2 f_{ij}\right), \quad i, j = 1, 2, \cdots, n-1.$$

The iterative method does not depend on the ordering of the equations and unknowns, but does depend on the index i and j.

2 (a) The results of one iteration are

$$\mathbf{x}_J \begin{pmatrix} 1/3 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \mathbf{x}_{GS} = \begin{pmatrix} 1/3 \\ 1/2 \\ 5/4 \end{pmatrix} \qquad \mathbf{x}_{SOR(1.5)} = \begin{pmatrix} 0 \\ 5/4 \\ 31/16 \end{pmatrix}$$

(b) and (c), The iteration matrices are:

$$R_J = \begin{bmatrix} 0 & 1/3 & -1/3 \\ 0 & 0 & -1/2 \\ 0 & 1/2 & 0 \end{bmatrix}, \qquad R_{GS} = \begin{bmatrix} 0 & 1/3 & -1/3 \\ 0 & 0 & -1/2 \\ 0 & 0 & -1/4 \end{bmatrix}.$$

Since $\|R_J\|_\infty = 2/3 < 1$ and $\|R_{GS}\|_\infty = 2/3 < 1$, both iterative methods converge.

3 For the first matrix, the eigenvalues of R are the diagonals. Notice that $|a_{ii}| < 1$ for $i = 1, 2, 3, 4$. We just need to check $|a_{55}| = 1 - \sin(\alpha\pi)$. Note that $0 < \sin x < 1$ if $0 < x < \pi$ and $\sin x$ is a periodic function of 2π. Thus, if $2k < \alpha < 2k + 1$, then the iterative method converges, where k is an integer.

For the second matrix we have $\|R\|_1 = 0.9999 < 1$, the iterative method converges.

4 (a) The iteration matrices are:

$$R_J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \qquad R_{G-S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{bmatrix}.$$

Since $\rho(R_J) = 2 > 1$ and $\rho(R_{GS}) = 4 > 1$, both iterative methods diverge.

(b) The matrix is weakly diagonally dominant and irreducible. Both Jacobi and Gauss-Seidel iterative methods converge.