Selected solutions

1 (a)

$$\frac{\|x_e - \bar{x}\|}{\|x_e\|} = \frac{\|A^{-1}b - A^{-1}(b+\delta b)\|}{\|x_e\|} \leq \frac{\|A^{-1}\|\|\delta b\|}{\|x_e\|}$$
$$\frac{\|A^{-1}\|\|A\|\|\delta b\|}{\|A\|\|x_e\|} \leq cond(A)\frac{\|\delta b\|}{\|b\|}.$$

(b) Also from the natural matrix norm, we have

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\|\|A^{-1}\| = cond(A) \tag{1}$$

2 (a) First we get $PA = LU$ decomposition. The total operation count is $O(2n^3/3)$. If we set $b = x^{(k)}$ and solve $Ax^{(j-1)} = x^{(j)}$, for $j = k, k-1, 1$ to form the recursive relation. The final $x^{(0)}$ is the solution. The total cost is $O(2n^3/3 + 2kn^2)$

(b) Let $x = A^{-1}b$ or $Ax = b$, after we have solved the equation to get x, then $\alpha = C^T x$. The total cost is about $O(2n^3/3 + 2n^2 + n)$.

3 The first matrix is strictly column diagonally dominant. It is not symmetric positive definite since $a_{11} < 0$.

The second matrix is weakly column diagonally dominant but not strictly. The matrix is symmetric positive definite since $det(A_1) = 2 > 0$, $det(A_2) = 3 > 0$, and $det(A_3) = det(A) = 4 > 0$.

For the third matrix to be strictly column diagonally dominant, the parameter should satisfy $|\alpha| > 1$ and $|\beta|| < 2$. If it is symmetric positive definite, then $\beta = -1$, $alf > 0$ and $2\alpha + \beta > 0$ which gives $\alpha > 1/2$.

3 Since A is symmetric, we have $\|A\|_2 = \max_i\{|\lambda_i(A)|\}$ and $\|A^{-1}\|_2 = \frac{1}{\min_i\{|\lambda_i(A)|\}}$. Thus we get $cond(A) = 300$. It is easy to check the x_1 is the solution since $r(x_1) = 0$; x_2 is not the solution so we can use the error estimate which is over-estimated!