

1. Let x_e be the solution of $Ax = b$ assuming that $\det(A) \neq 0$, \bar{x} be the solution of $Ax = b + \delta b$, show that

$$\frac{\|x_e - \bar{x}\|}{\|x_e\|} \leq \text{cond}(A) \frac{\|\delta b\|}{\|b\|}.$$

Hint: $\|b\| = \|AA^{-1}b\| \leq \|A\|\|A^{-1}b\|$. Show also that $\text{cond}(A) \geq 1$.

2. Given an n -by- n non-singular matrix A , how do you efficiently solve the following problems using Gaussian elimination with partial column pivoting?
- Solve $A^k x = b$, where k is a positive integer.
 - Computer $\alpha = c^T A^{-1}b$, where c and b are two vectors.

You should (1) describe your algorithm; (2) present a pseudo-code; (3) find out the required operation counts.

3. Check whether the following matrices are:

- Strictly column diagonally dominant.
- Symmetric positive definite.

Justify your conclusion. What is the significance of knowing these special matrices to the Gaussian related algorithms? Answer this question by considering issues of accuracy, speed, and the storage.

$$\begin{bmatrix} -5 & 2 & 1 & 0 \\ 2 & 7 & -1 & -1 \\ 1 & -1 & 5 & 1 \\ 0 & -1 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} \alpha & \beta \\ -1 & 2 \end{bmatrix}$$

Find the Cholesky decomposition $A = LL^T$ or $A = LDL^T$ for the middle matrix. Note that, you need to determine the range of α and β .

4. Given a matrix A and a vector b

$$A = \begin{bmatrix} \frac{1}{100} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{100} \\ 1 \\ 1 \end{bmatrix}$$

- Find the condition number of A in 2-norm.
- Given $x_1 = [1 \ 1 \ 1]^T$ and $x_2 = [1.1 \ 1 \ 0.9]^T$, find the residual, and the norm of the residual, for both x_1 and x_2 in 2-norm.
- Is one of x_1 and x_2 the exact solution of the system $Ax = b$?

- (d) If x_1 or x_2 is not the solution, use the error estimate (3.42) on page 47 of the notes (relation between the residual and the relative error) to give an estimate error bound for the relative error.
- (e) Find the actual relative error for x_1 and x_2 as approximations to $Ax = b$ and compare with the error bound that you just have got. How much is the difference?

5. (Programming Part) Let A be a symmetric positive definite matrix.

- (a) Derive the algorithms for $A = LDL^T$ decomposition, where L is a unit lower triangular matrix, and D is a diagonal matrix.
- (b) Write a Matlab code (or other language if you prefer) to do the factorization and solve the linear system of equations $Ax = b$ using the factorization. **Hint:** the process is the following:

$$\begin{aligned} Ly &= b, & y \text{ is the unknown,} \\ Dz &= y, & z \text{ is the unknown,} \\ L^T x &= z, & x \text{ is the unknown, which is the solution.} \end{aligned}$$

Construct at least one example that you know the exact solution to validate your code. (Talk to the instructor if you need help on this.)

Selected solutions

1 (a)

$$\frac{\|x_e - \bar{x}\|}{\|x_e\|} = \frac{\|A^{-1}b - A^{-1}(b + \delta b)\|}{\|x_e\|} \leq \frac{\|A^{-1}\|\|\delta b\|}{\|x_e\|}$$

$$\frac{\|A^{-1}\|\|A\|\|\delta b\|}{\|A\|\|x_e\|} \leq \mathit{cond}(A) \frac{\|\delta b\|}{\|b\|}.$$

(b) Also from the natural matrix norm, we have

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\|\|A^{-1}\| = \mathit{cond}(A) \quad (1)$$

2 (a) First we get $PA = LU$ decomposition. The total operation count is $O(2n^3/3)$. If we set $b = x^{(k)}$ and solve $Ax^{(j-1)} = x^{(j)}$, for $j = k, k-1, 1$ to form the recursive relation. The final $x^{(0)}$ is the solution. The total cost is $O(2n^3/3 + 2kn^2)$

(b) Let $x = A^{-1}b$ or $Ax = b$, after we have solved the equation to get x , then $\alpha = C^T x$. The total cost is about $O(2n^3/3 + 2n^2 + n)$.

3 The first matrix is strictly column diagonally dominant. It is not symmetric positive definite since $a_{11} < 0$.

The second matrix is weakly column diagonally dominant but not strictly. The matrix is symmetric positive definite since $\det(A_1) = 2 > 0$, $\det(A_2) = 3 > 0$, and $\det(A_3) = \det(A) = 4 > 0$.

For the third matrix to be strictly column diagonally dominant, the parameter should satisfy $|\alpha| > 1$ and $|\beta| < 2$. If it is symmetric positive definite, then $\beta = -1$, $af > 0$ and $2\alpha + \beta > 0$ which gives $\alpha > 1/2$.

3 Since A is symmetric, we have $\|A\|_2 = \max_i\{|\lambda_i(A)|\}$ and $\|A^{-1}\|_2 = \frac{1}{\min_i\{|\lambda_i(A)|\}}$. Thus we get $\mathit{cond}(A) = 300$. It is easy to check the x_1 is the solution since $r(x_1) = 0$; x_2 is not the solution so we can use the error estimate which is over-estimated!