Selected Solutions and Comments

1 (b) $\|x\|_1 = |\sin k| + |\cos k| + 2^k$. $\|x\|_2 = \sqrt{1+2^{4k}}$. $\|x\|_\infty = 2^k$ since $2^k \geq 1$.

2 For the first matrix, we have $\|A\|_1 = 5$, $\|A\|_\infty = 3$. $det(\lambda I - A^A) = \lambda^2 - 14\lambda + 9 = 0$. The two roots are $7 \pm \sqrt{40}$. Thus $\|A\|_2 = \sqrt{7+2\sqrt{10}}$.

For the second matrix, we have $\|A\|_1 = \|A\|_\infty = 3$. Since A is symmetric, $\|A\|_2 = \max|\lambda_i(A)| = 3$.

Proof:

$$\|AB\| = \max_{\|\mathbf{x}\|\neq 0} \frac{\|AB\mathbf{x}\|}{\|\mathbf{x}\|} \leq \max_{\|\mathbf{x}\|\neq 0} \frac{\|A\|\,\|B\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|A\| \max_{\|\mathbf{x}\|\neq 0} \frac{\|B\mathbf{x}\|}{\|\mathbf{x}\|} = \|A\|\|B\| \quad (1)$$

where we have used the inequality $\|AB\mathbf{x}\| \leq \|A\|\|B\mathbf{x}\|$ since $B\mathbf{x}$ is a vector.

$$\|QA\|_2 = \max_{\|\mathbf{x}\|\neq 0} \frac{\|QA\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{x}\|\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \|A\|_2 \quad (2)$$

where we have used the equality $\|QA\mathbf{x}\|_2 = \|A\mathbf{x}\|_2$ since $A\mathbf{x}$ is a vector.

4 You should use the compact form (one matrix) and partial column pivoting. We get

$$P = P_{12}P_{23}P_{34}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 0 & 1/3 & 5/7 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & \frac{28}{9} & -4/9 \\ 0 & 0 & 0 & \frac{24}{7} \end{bmatrix}$$

Thus, $det(A) = (-1)^3 det(U) = -96$. To solve the linear system of equations, we use forward substitution to solve $L\mathbf{y} = p^T\mathbf{b}$ to get $\mathbf{y} = [6\ 2\ 8/3\ 24/7]^T$. Final we solve $U\mathbf{x} = \mathbf{y}$ to get $\mathbf{x} = [1\,1\,1\,1]^T$.

5 Let $B = [b_1, b_2, \cdots, b_m]$. To solve the X, we set $Ly_i = Pb_i$, and then $Ux_i = y_i$ for $i = 1, 2, \cdots, m$. So the total number of arithmetics $\pm, \times, \div$ is $O(2n^3/3 + 2n^2m)$.

For the second method, we need to use the forward/backward substitution to get A^{-1} first. The i-th column of A^{-1} can be obtained by solving $Az_i = e_i$. The total number of arithmetics $\pm, \times, \div$ needed to get A^{-1} is $O(2n^3/3 + n^2n)$. Note that, we just need half of the operations when we use the forward substitution due to the zeros in e_i. It is equivalent to perform the Gaussian elimination to the augmented matrix $[A\ I]$ then apply the backward substitution. In the second method, after we get A^{-1}, we need to perform the matrix vector multiplication $A^{-1}B$ which requires $O(2n^2m)$ operations. Thus the total cost is $O(4/3n^3 + 2n^2m)$ which is more than the fist algorithm especially when $m \ll n$.

6 See page 3-5 of the notes for the derivation. Note the Gibbs phenomenon (oscillations) near the boundary. That is why we do not use high order polynomial interpolations to avoid oscillations. Piecewise polynomials, also called splines are preferred.