

1. Find $\|x\|_p$, $p = 1, 2, \infty$ for the following vectors

(a) $\mathbf{x} = (3, -4, 0, -3/2)^T$.

(b) $\mathbf{x} = (\sin k, \cos k, 2^k)^T$ for a fixed positive integer k .

(c) $\mathbf{x} = (4/(k+1), -2/k^2, k^2 e^{-k})^T$ for a fixed positive integer k .

2. Find $\|A\|_p$, $p = 1, 2, \infty$ for the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}; \quad \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix};$$

3. (a) Show that $\|x\|_\infty$ is equivalent to $\|x\|_2$. That is to find constants C and c such that $c \leq \|x\|_\infty \leq \|x\|_2 \leq C\|x\|_\infty$. Note that you need to determine such constants that the equalities are true for some particular x .

(b) Show that $\|AB\| \leq \|A\|\|B\|$ for any natural matrix norm, and $\|QA\|_2 = \|A\|_2$ if Q is an orthogonal matrix ($Q^H Q = I$, $Q Q^H = I$).

4. Given

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$$

(a) Use Gaussian elimination with the partial pivoting to find the matrix decomposition $PA = LU$. This is a paper problem and you are asked to use exact calculations (use fractions if necessary).

(b) Find the determinant of the matrix A .

(c) Use the factorization to solve $Ax = b$.

5. Consider solving $AX = B$ for X , $A \in R^{n,n}$, $B \in R^{n,m}$. There are two obvious algorithms. The first one is to get $A = PLU$ using Gaussian elimination, and then to solve for each column of X by forward and backward substitution. The second algorithm is to compute A^{-1} using Gaussian elimination and then to multiply $A^{-1}B$ to get X . Count the number of operations by each algorithm and determine which one is faster.

6. (Programming Part) Given a sequence of data

$$(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m), (x_{m+1}, y_{m+1}),$$

write a program to interpolate the data using the following model

$$y(x) = a_0 + a_1 x + \dots + a_{m-1} x^{m-1} + a_m x^m.$$

- (a) Derive the linear system of equations for the interpolation problem.
- (b) Let $x_i = (i - 1)h$, $i = 1, 2, \dots, m + 1$, $h = 1/m$, $y_i = \sin \pi x_i$, write a computer code using the Gaussian elimination with column partial pivoting to solve the problem. Test your code with $m = 4, 8, 16, 32, 64$ and plot the error $|y(x) - \sin \pi x|$ with 100 or more points between 0 and 1, that is, predict the function at more points in addition to the sample points. For example, you can set $h = 1/100$; $x = 0 : h : 1$, $y1(i) = a_0 + a_1 x(i) + \dots + a_{m-1} (x(i))^{m-1} + a_m (x(i))^m$, $y2(i) = \sin(\pi x(i))$, `plot(x1, y1 - y2)`.
- (c) Record the CPU time (in Matlab type `help cputime`) for $m = 50, 100, 150, 200, \dots, 350, 400$. Plot the CPU time versus m . Then use the Matlab function `polyfit` $z = \text{polyfit}(m, \text{cputime}(m), 3)$ to find a cubic fitting of the CPU time versus m . Write down the polynomial and analyze your result. Does it look like a cubic function?

7. **Extra Credit:** Choose **one** from the following (Note: please do not ask the instructor about the solution since it is extra credit):

- (a) Let $A \in R^{n \times n}$. Show that $\|A\|_2 = \max_{1 \leq i \leq n} \sqrt{\lambda_i(A^T A)}$, where $\lambda_i(A^T A)$, $i = 1, 2, \dots, n$ are the eigenvalues of $A^T A$.
- (b) Show that if A is a symmetric positive definite matrix, then after one step of Gaussian elimination (without pivoting), then reduced matrix A_1 in

$$A \implies \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix}$$

must be symmetric positive definite. Therefore no pivoting is necessary.