MA580/CSC580 Review

Please also see Mid-term Review; Homework Assignments; Short Proofs and Questions asked in class.

Concepts:

Iteration matrix; Spectral radius $\rho(A)$; eigenvalues and eigenvectors; diagonalizable matrix; Residual Vector. the least squares solution, Householder matrix; curve fitting by polynomials or by other base functions; over-determined/under-determined system; Gershgorin circles; Eigenvalues and Eigenvectors; Similarity transformation; Orthogonal matrices; The dominant/(least dominant) eigenvalue; Gerschgorin circles; Householder transformation; Upper Hessenberg matrices; Least squares solutions; Singular value decomposition (SVD); The normal equations; Pseudo-inverse

Stationary Iterative Algorithms: $x^{(k)} = T\,x^{(k-1)} + c, \quad k = 1, 2, \cdots$

Jacobi Iterative Method: $T_g = D^{-1}(L+U); \quad c = D^{-1}b.$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \ i = 1, 2, \cdots, n.$$

The Jacobi method converges if A is strictly row diagonally dominant.

Gauss-Seidel Iterative Method: $T_g = (D-L)^{-1}U; \quad c = (D-L)^{-1}b.$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \ i = 1, 2, \cdots, n.$$

The Jacobi method converges if A is strictly row diagonally dominant or symmetric positive definite.

SOR Iterative Method: $T_\omega = (D-\omega L)^{-1}[(1-\omega)D + \omega U]; \quad c = \omega(D-\omega L)^{-1}b.$

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right], \ i = 1, 2, \cdots, n.$$

The SOR method converges if A is symmetric positive definite and $0 < \omega < 2$.

The Power method:

$$\begin{aligned} y_{k+1} &= Ax_k \\ x_{k+1} &= \frac{y_{k+1}}{\|y_{k+1}\|_2} \\ \mu_{k+1} &= x_{k+1}^T A x_{k+1} \end{aligned}$$

The shifted inverse Power method to find a specific eigenvalue λ_k, if $q \approx \lambda_k$

$$\begin{aligned} (A - qI)\, y_{k+1} &= x_k \\ x_{k+1} &= \frac{y_{k+1}}{\|y_{k+1}\|_2} \\ \mu_{k+1} &= x_{k+1}^T A x_{k+1} \end{aligned}$$

QR method and shifted QR method

$$\begin{aligned} A_k - q_k I &= Q_k R_k \\ A_{k+1} &= R_k Q_k + q_k I \end{aligned}$$

Theories:

- Error estimates of the direct methods and iterative methods.
- Show that the iterative method will converge if $\|R\| < 1$. Derive the convergence speed.
- Show that $\rho(A) \leq \|A\|$.
- Orthogonal matrices ($Q^{-1} = Q^T,\ Q^T Q = I,\ ||Qx||_2 = ||x||_2,\ ||QA||_2 = ||A||_2,\ \kappa_2(QA) = \kappa_2(A)$ etc.)
- The Householder transformation theorem ($Px = y$), How to choose y (say $y = \alpha e_1$, need to find α, pay attention to the sign!) and w, $w^T w = 1$ to form $P = I - 2\, w\, w^T$.

Supplementary exercises

1. Homework problems; problems from class; class notes etc.
2. What is the essential condition for the convergence of the Power method or the symmetric Power method? (The dominant eigenvalue is a simple eigenvalue).
3. What is the convergence speed of the Power method?
4. Can we use the Householder *similarity transformations* to reduce a matrix A to a diagonal matrix?
5. What is the relation between the eigenvalues and eigenvectors of A and the eigenvalues and eigenvectors of
   (a) A^{-1}?
   (b) $A - q\, I$?
   (c) $(A - q\, I)^{-1}$?
6. If λ is an eigenvalue of a matrix A,

(a) show that $|\lambda| \le ||A||$ and so $\rho(A) \le ||A||$;

(b) use Gerschgorin theorem to show that $|\lambda| \le ||A||_\infty$.

7. Show that $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -5 \end{bmatrix}$ has three distinct eigenvalues $|\lambda_1| > |\lambda_2| > |\lambda_3|$. In order to approximate the eigenvalue λ_2, apply the first two iterations of the inverse Power method to the shifted matrix $A - 3I$.

8. Let

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Use above to solve the system of equations $Ax = b$ where $b = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$.

9. Let $A \in R^{m \times n}$ with $rank(A) = n$. We can use the QR method or the Cholesky decomposition for the normal equation $A^T Ax = A^T b$ to solve the least squares problem $Ax = b$. Compare the two methods and make a recommendation. **Hint:** Consider the operation account, the storage, accuracy etc.

10. Given the following linear system of equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ 2x_2 + x_3 &= 2 \\ -x_2 + 2x_3 &= 2 \end{aligned}$$

(a) With $x^{(0)} = [1, -1, 1]^T$, find the *first* and *second* iteration of the Jacobi, Gauss-Seidel, and SOR ($\omega = 1.5$) methods.

(b) Write down the Jacobi and Gauss-Seidel iteration matrices R_J and R_{GS}.

(c) Do the Jacobi and Gauss-Seidel iterative methods converge?

(d) Use the QR method and the normal equation to solve the least squares problem:

$$A = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}; \quad A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

where $b = \begin{bmatrix} 3 & 0 & 0 & 6 & -8 \end{bmatrix}^T$. Find the 2-norm of the residual vector corresponding to the least squares solution.

11. Assume we use the following iterative method to solve $Ax = b$, where A is a non-singular arbitrary matrix,

$$\begin{aligned} x_k &= x_{k-1} + \alpha_k p_k \\ e_k &= e_{k-1} - \alpha_k p_k, & \qquad e_k = A^{-1}b - x_k, \\ r_k &= r_{k-1} - \alpha_k A p_k, & \qquad r_k = b - Ax_k, \end{aligned}$$

for given p_k. Determine α_k according to the following:

(a) Minimize $\|e_k\|_2$.

(b) Minimize $\|r_k\|_2$.

(c) Which approach is computable? For the second case, also show that $r_k^T A p_k = 0$.

Hint: Use the fact that $\|x\|_2^2 = x^T x$.