

# Supplementary to the Notes of MA402

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## Ch.1.2.3– Ch.6

- Ignore the last two paragraphs after equation (3) in Ch.1.2.3.
- The matrix vector form of the system of equations (1) is

$$\mathbf{u}^{k+1} = A \mathbf{u}^k + \mathbf{b}^k$$

where  $A$ ,  $\mathbf{u}^k$ , and  $\mathbf{b}$  are

$$A = \begin{bmatrix} 1-2\alpha & \alpha & 0 & \cdots & 0 \\ \alpha & 1-2\alpha & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \alpha & 1-2\alpha & \alpha \\ 0 & \cdots & 0 & \alpha & 1-2\alpha \end{bmatrix}, \quad \mathbf{u}^k = \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{m-2}^k \\ u_{m-1}^k \end{bmatrix}, \quad \mathbf{b}^k = \frac{1}{\rho c} \begin{bmatrix} f_1^k + \alpha u(0, t^k) \\ f_2^k \\ \vdots \\ f_{m-2}^k \\ f_{m-1}^k + \alpha u(L, t^k) \end{bmatrix}$$

where  $f_i^k = f(x_i, x^k)$  and we assume that

$$x_i = i h, \quad i = 0, 1, \dots, m, \quad x_0 = 0, \quad x_m = L, \quad h = \frac{1}{L}.$$

- Similarly, the matrix in page 2 of Ch. 1.3 has similar structure as above with the diagonal element being  $1 - 2\alpha - d$ .
- The matrix vector form of the system of equations (1) in page 3 of Ch.1.4.2 should be  $\mathbf{u}^{k+1} = A \mathbf{u}^k + \mathbf{b}^k$ , where  $A$ ,  $\mathbf{u}^k$ , and  $\mathbf{b}^k$  are

$$A = \begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ d & c & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & d & c & 0 \\ 0 & \cdots & 0 & d & c \end{bmatrix}, \quad \mathbf{u}^k = \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{m-1}^k \\ u_m^k \end{bmatrix}, \quad \mathbf{b}^k = \begin{bmatrix} d u(0, t^k) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Note the solution  $u_m^k$  at the right boundary  $x = L$  is also an unknown.

## Matrix and vector norms, see also Ch.3.5

A vector norm for  $\mathbf{x} \in R^n$  is a special function that satisfies the following conditions

1.  $f(\mathbf{x}) = f(x_1, x_2, \dots, x_n) \geq 0$ ;  $f(\mathbf{x}) = 0$  if and only  $\mathbf{x} = 0$ .
2.  $f(\alpha\mathbf{x}) = |\alpha|f(\mathbf{x})$ .
3.  $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ .

Such function is denoted as  $\|\mathbf{x}\|$ .

### Example of vector norms

1.  $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \dots + |x_n| = \sum_{i=1}^n |x_i|$
2.  $\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2} = \sqrt{\sum_{i=1}^n |x_i|^2}$ .
3.  $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\} = \max_{1 \leq i \leq n} |x_i|$ .
4.  $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p} = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$  for  $p > 0$ .

### Example of matrix norms $A \in R^{m \times n}$

1.  $\|A\|_1 = \max \left\{ \sum_{i=1}^m |a_{i1}|, \sum_{i=1}^m |a_{i2}|, \dots, \sum_{i=1}^m |a_{in}|, \right\} = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$
2.  $\|A\|_\infty = \max \left\{ \sum_{j=1}^n |a_{1j}|, \sum_{j=1}^n |a_{2j}|, \dots, \sum_{j=1}^n |a_{mj}|, \right\} = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$
3.  $\|A\|_2 = \max_{1 \leq i \leq n} \sqrt{\lambda_i(A^T A)}$ , where  $\lambda_i(A^T A)$  are the eigenvalues of  $A^T A$ .

Example of vector and matrix norm:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ -6 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -4 & 1 & 2 \\ 0 & 4 & -2 & 1 \\ 7 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Then it is easy to get

$$\begin{aligned} \|\mathbf{x}\|_1 &= 14, & \|\mathbf{x}\|_2 &= \sqrt{62}, & \|\mathbf{x}\|_\infty &= 6. \\ \|A\|_1 &= 11, & \|A\|_\infty &= 10. \end{aligned}$$

The matrix 2-norm is more difficult to compute directly, using Matlab  $\text{norm}(A, 2)$  we can get  $\|A\|_2 = 8.2325$ . Another example, the matrix of the system of equations (1) in page 3 of Ch.1.2. with  $\alpha = 0.2$  and  $m = 5$ .

$$A = \begin{bmatrix} 3/5 & 1/5 & 0 & 0 & 0 \\ 1/5 & 3/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 3/5 & 1/5 \\ 0 & 0 & 0 & 1/5 & 3/5 \end{bmatrix}$$

We have  $\|A\|_1 = \|A\|_\infty = 1$ , and  $\|A\|_2 = 0.9464 < 1$ . So the algorithm converges and the steady state solution exists.

**Theorem 1** *Given a matrix  $A \in R^{n \times n}$ , we have the following*

1. *If there is one matrix norm such that  $\|A\| < 1$ , then  $\lim_{k \rightarrow \infty} A^k = \mathbf{0}^{n \times n}$ .*
2.  *$\lim_{k \rightarrow \infty} A^k = \mathbf{0}^{n \times n}$  if and only if  $\max |\lambda_i(A)| < 1$ .*

Note that if  $\|A\|_1 > 1$ , or  $\|A\|_2 > 1$ , or  $\|A\|_\infty > 1$ , we can not claim anything on the convergence of  $\lim_{k \rightarrow \infty} A^k$  which means it is *inclusive*.