Runge-Kutta Method

Purpose: High order accurate one-step, multi-stage method. Provide a criteria for adaptive time step size.

1 General Algorithm of RK-k:

$$y_{i+1} = y_i + h\left(c_0 f_0 + c_1 f_1 + \cdots + c_k f_k\right) \quad (1)$$

Example: Improved Euler's method is one of RK2 methods:

$$f_0 = f(x_i, y_i), \qquad f_1 = f(x_{i+1}, y_i + hf_0),$$
$$y_{i+1} = y_i + h\left(\frac{1}{2}f_0 + \frac{1}{2}f_1\right).$$

Usually

$$\begin{aligned}
f_0 &= f(x_i, y_i) \\
f_1 &= f\left(x_i + \alpha_1 h, y_i + h\beta_{10} f_0\right) \\
\cdots & \quad \cdots \\
f_k &= f\left(x_i + \alpha_k h, y_i + h(\beta_{k0} f_0 + \beta_{k1} f_1 + \cdots + \beta_{k,k-1} f_{k-1}\right)
\end{aligned}$$

i	α_i	β_{ij}	c_i
0	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$			

One of RK2 method –Improved Euler's method:

i	α_i	β_{ij}	c_i
0	0·		$\frac{1}{2}$
1	1 ·	1	$\frac{1}{2}$

One of RK4 method:

i	α_i	β_{ij}	c_i
0	·		$\frac{1}{6}$
1	$\frac{1}{2}$ ·	$\frac{1}{2}$	$\frac{1}{3}$
2	$\frac{1}{2}$ ·	0 $\frac{1}{2}$	$\frac{1}{3}$
3	1 ·	0 0 1	$\frac{1}{6}$

It is equivalent to:

$$\begin{aligned}
f_0 &= f(x_i, y_i), \\
f_1 &= f(x_i + \frac{h}{2}, y_i + \frac{h}{2} f_0), \\
f_2 &= f(x_i + \frac{h}{2}, y_i + \frac{h}{2} f_1), \\
f_3 &= f(x_i + h, y_i + h f_2) \\
y_{i+1} &= y_i + \frac{h}{6}\left(f_0 + 2f_1 + 2f_2 + f_3\right).
\end{aligned} \tag{2}$$

2 Fehlberg's RKF4(5) Method

Two Runge-Kutta methods, RK4+RK5. The purpose of the method is to choose suitable time step.

i	α_i	β_{ij}					c_i	$\hat{c}_i$
0 ·	0						$\frac{1}{9}$	$\frac{47}{450}$
1 ·	$\frac{2}{9}$	$\frac{2}{9}$					0	0
2 ·	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{4}$				$\frac{9}{20}$	$\frac{12}{25}$
3 ·	$\frac{3}{4}$	$\frac{69}{128}$	$-\frac{243}{128}$	$\frac{135}{64}$			$\frac{16}{45}$	$\frac{32}{225}$
4 ·	1	$-\frac{17}{12}$	$\frac{27}{4}$	$-\frac{27}{5}$	$\frac{16}{15}$		$\frac{1}{12}$	$\frac{1}{30}$
5 ·	$\frac{5}{6}$	$\frac{65}{432}$	$-\frac{5}{16}$	$\frac{13}{16}$	$\frac{4}{27}$	$\frac{5}{144}$	0	$\frac{6}{25}$

$$\begin{aligned} \text{RK4: } \bar{y}_{i+1} &= y_i + h\left(c_0 f_0 + c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4\right) \\ \text{RK5: } y_{i+1} &= y_i + h\left(\hat{c}_0 f_0 + \hat{c}_1 f_1 + \hat{c}_2 f_2 + \hat{c}_3 f_3 + \hat{c}_4 f_4 + \hat{c}_5 f_5\right) \end{aligned} \tag{3}$$

A typical expression for f_i

$$f_4 = f\left(x_i + h, y_i + h\left[-\frac{17}{12}f_0 + \frac{27}{4}f_1 - \frac{27}{5}f_2 + \frac{16}{15}f_3\right]\right)$$

Can prove that

$$|\bar{y}_i - y_i| = Ah^5. \tag{4}$$

Use this to determine the time step.

- $|\bar{y}_i - y_i| > tol$, use a smaller time step.
- $|\bar{y}_i - y_i| < tol$, use a larger time step.