CORRECTION TO: "A COMPARISON OF THE EXTENDED FINITE ELEMENT METHOD WITH THE IMMERSED INTERFACE METHOD . . . " [CAMCOS 1 (2006), 207–228] *

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Abstract. A recent paper by Vaughan, Smith, and Chopp [Comm. App. Math. & Comp. Sci. 1 (2006), 207-228] reported numerical results for three examples using the immersed interface method (IIM) and the extended finite element method (X-FEM). The results presented for the IIM showed first-order accuracy for the solution and inaccurate values of the normal derivative at the interface. This was due to an error in the implementation. The purpose of this note is to present correct results using the IIM for the same examples used in that paper, which demonstrate the expected second-order accuracy in the maximum norm over all grid points. Results now indicate that on these problems the IIM and X-FEM methods give comparable accuracy in solution values. With appropriate interpolation it is also possible to obtain nearly second order accurate values of the solution values and normal derivative at the interface with the IIM.

Key words. Immersed Interface Method (IIM), Elliptic interface problems, finite difference methods, discontinuous coefficients, singular source term, convergence order

AMS subject classifications. 65N06, 65N50

1. Introduction. The immersed interface method (IIM) [3, 5, 4] is a method for solving PDE's with discontinuous coefficients and singular sources at an interface using values at grid points. For elliptic problems, uniform second-order accuracy is obtained at all grid points, including those near and on the interface, by correcting the truncation error at the interface to first order using jump conditions. This second-order accuracy has been well established in computational practice and theoretical analysis; see for example, [1, 2, 3, 4, 5].

The paper [6] presented numerical results for three test problems using the IIM as well as the extended finite element method. The reported results with the IIM showed only first-order accuracy for the solution and inaccurate values of the normal derivative at the interface. This was due to error in the implementation.

The purpose of this note is to present correct versions of the Tables in [6]. These results confirm that the IIM gives second-order accuracy in the maximum norm at the grid points for all the examples. Values at the interface are obtained to second order by interpolation. Furthermore, the normal derivative of the solution at the interface, also interpolated from values at grid points, is also nearly second order accurate. This agrees with the theoretical prediction of Beale and Layton [1].

The corrected results indicate that on these problems the IIM and X-FEM methods give comparable accuracy in solution values, although the IIM and the X-FEM results are no longer solved using the same interpolation scheme for locating the interface from a discrete implicit representation. The IIM results presented here are using a cubic spline reconstruction, while the X-FEM results are from using a linear

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interpolant. The IIM with appropriate interpolation also gives much more accurate normal derivatives at the interface than those reported for X-FEM in [6]. However, as noted in [7], X-FEM solutions can be significantly improved by using more specialized enrichment functions. These extra enrichment functions were not used for the results presented here.

The computer codes used are available on request.

2. Corrected Tables. The examples in the paper [6] are taken from the original IIM paper [3] and the tables in [3] are correct and indicate second order accuracy. For this note we have repeated these experiments using a new version of the IIM developed in 2001 by Li and Ito [4]. With the new code we have also computed the error in the normal derivative of the solution at the interface for corrected versions of Tables 4, 7, and 10. These errors were not presented in the original paper [3].

The problems and notation are presented in [6] and not repeated here. The error displayed in the tables is the maximum error over all grid points,

$$\|E_n\|_\infty = \max_{i,j} u(x_i, y_j) - u_{ij},$$

where u_{ij} is the computed approximation at the grid point (x_i, y_j). In [6] this was denoted by T_n, but we are now being consistent with the notation of LeVeque and Li [3] where T_n is used to denote the truncation error rather than the global error. The statement made at the bottom of page 219 in [6] about the notation of LeVeque and Li is incorrect. We use E_n^I to denote the error at the interface (when appropriate interpolation is used), and E_n^D to denote the error in the normal derivative at the interface.

The ratios of successive errors, $\|E_n\|_\infty / \|E_{2\,n}\|_\infty$, are displayed to show the order of convergence. A ratio of 2 corresponds to first order accuracy, while a ratio of 4 indicates second order accuracy. Note that we do not necessarily expect ratios of exactly 4 since the accuracy seen on a particular grid is sensitive to how the interface cuts between the grid points.

Below are the corrected tables that should be inserted in [6] in place of the tables presented there. We give new versions of all tables except Table 2, which concerned the system size rather than the errors. We have extended the tables to $n = 640$ for the IIM to better demonstrate the asymptotic rate of convergence.

n	Step Enrichment		Ramp Enrichment		n	IIM	
	$\|E_n\|_\infty$	ratio	$\|E_n\|_\infty$	ratio		$\|E_n\|_\infty$	ratio
19	3.8397×10^{-3}		7.8138×10^{-3}		20	3.4796×10^{-3}	
39	9.3782×10^{-4}	4.0943	3.9577×10^{-3}	1.9743	40	9.6364×10^{-4}	3.6109
79	2.3034×10^{-4}	4.0715	1.9029×10^{-3}	2.0798	80	2.0531×10^{-4}	4.6935
159	6.4061×10^{-5}	3.5956	9.3797×10^{-4}	2.0287	160	6.0650×10^{-5}	3.3852
319	1.5619×10^{-5}	4.1015	4.7646×10^{-4}	1.9686	320	1.5702×10^{-5}	3.8626
					640	3.7712×10^{-6}	4.1636

Table 1: Errors at grid points for example #1

n	Step Enrichment		Ramp Enrichment		n	IIM	
	$\|E_n^I\|_\infty$	ratio	$\|E_n^I\|_\infty$	ratio		$\|E_n^I\|_\infty$	ratio
19	5.1857×10^{-3}		2.1871×10^{-2}		20	3.6675×10^{-3}	
39	1.2444×10^{-3}	4.1672	1.1708×10^{-2}	1.8680	40	9.9843×10^{-4}	3.6733
79	3.0043×10^{-4}	4.1421	6.0996×10^{-3}	1.9482	80	2.0727×10^{-4}	4.8172
159	8.8146×10^{-5}	3.4083	3.1101×10^{-3}	1.9612	160	6.1075×10^{-5}	3.3936
319	1.9315×10^{-5}	4.5636	1.6142×10^{-3}	1.9267	320	1.5816×10^{-5}	3.8615
					640	3.7840×10^{-6}	4.1799

Table 3: Interface errors for example #1

n	Step Enrichment		Ramp Enrichment		n	IIM	
	$\|E_n^D\|_\infty$	ratio	$\|E_n^D\|_\infty$	ratio		$\|E_n^D\|_\infty$	ratio
19	4.1828×10^{-1}		1.8292×10^{-0}		20	3.8809×10^{-3}	
39	1.6067×10^{-1}	2.6033	1.6479×10^{-0}	1.1100	40	1.6169×10^{-3}	2.4002
79	9.3826×10^{-2}	1.7124	1.3096×10^{-0}	1.2583	80	3.8618×10^{-4}	4.1870
159	4.5301×10^{-2}	2.0712	1.4733×10^{-0}	0.8889	160	1.5042×10^{-4}	2.5673
319	2.2290×10^{-2}	2.0323	1.3818×10^{-0}	1.0662	320	4.0937×10^{-5}	3.6744
					640	1.0109×10^{-5}	4.0497

Table 4: Interface derivative errors for example #1

n	Step Enrichment		n	IIM	
	$\|E_n\|_\infty$	ratio		$\|E_n\|_\infty$	ratio
19	1.7613×10^{-3}		20	1.7445×10^{-3}	
39	4.1771×10^{-4}	4.2166	40	4.8638×10^{-4}	3.5867
79	1.0289×10^{-4}	4.0598	80	1.4476×10^{-4}	3.3598
159	3.0164×10^{-5}	3.4110	160	3.0120×10^{-5}	4.8063
319	6.7960×10^{-6}	4.4385	320	8.2255×10^{-6}	3.6618
			640	2.0599×10^{-6}	3.9932

Table 5: Errors at grid points for example #2

n	Step Enrichment		n	IIM	
	$\|E_n^I\|_\infty$	ratio		$\|E_n^I\|_\infty$	ratio
19	1.6517×10^{-3}		20	1.6370×10^{-3}	
39	3.3824×10^{-4}	4.8832	40	4.5883×10^{-4}	3.5678
79	8.2238×10^{-5}	4.1129	80	1.2910×10^{-4}	3.5540
159	3.1568×10^{-5}	2.6051	160	2.5791×10^{-5}	5.0058
319	7.4612×10^{-6}	4.2310	320	6.7347×10^{-6}	3.8295
			640	1.5924×10^{-6}	4.2292

Table 6: Interface errors for example #2

n	Step Enrichment		n	IIM	
	$\|E_n^D\|_\infty$	ratio		$\|E_n^D\|_\infty$	ratio
19	2.7307×10^{-1}		20	1.0232×10^{-2}	
39	1.2776×10^{-1}	2.1374	40	5.2912×10^{-3}	1.9337
79	6.1203×10^{-2}	2.0875	80	3.6363×10^{-3}	1.4551
159	4.8216×10^{-2}	1.2694	160	9.3114×10^{-4}	3.9052
319	2.4790×10^{-2}	1.9450	320	2.7492×10^{-4}	3.3869
			640	6.8856×10^{-5}	3.9927

Table 7: Interface derivative errors for example #2

n	Step Enrichment		n	IIM	
	$\|E_n\|_\infty$	ratio		$\|E_n\|_\infty$	ratio
19	1.7648×10^{-4}		20	4.37883×10^{-4}	
39	6.0109×10^{-5}	2.9360	40	1.07887×10^{-4}	4.0587
79	1.7769×10^{-5}	3.3828	80	2.77752×10^{-5}	3.8843
159	4.8626×10^{-6}	3.6542	160	7.49907×10^{-6}	3.7038
319	1.2362×10^{-6}	3.9335	320	1.74001×10^{-6}	4.3098
			640	4.50600×10^{-7}	3.8614

Table 8: Errors at grid points for example #3

n	Step Enrichment		n	IIM	
	$\|E_n^I\|_\infty$	ratio		$\|E_n^I\|_\infty$	ratio
19	4.7842×10^{-4}		20	2.7671×10^{-3}	
39	1.0659×10^{-4}	4.4884	40	4.5255×10^{-4}	6.1143
79	2.8361×10^{-5}	3.7583	80	7.7651×10^{-5}	5.8280
159	7.3603×10^{-6}	3.8532	160	1.3988×10^{-5}	5.5512
319	2.0634×10^{-6}	3.5671	320	2.7647×10^{-6}	5.0596
			640	6.5488×10^{-7}	4.2217

Table 9: Interface errors for example #3

n	Step Enrichment		n	IIM	
	$\|E_n^D\|_\infty$	ratio		$\|E_n^D\|_\infty$	ratio
19	5.6520×10^{-2}		20	3.8100×10^{-3}	
39	2.4190×10^{-2}	2.3365	40	1.7600×10^{-3}	2.1648
79	9.4512×10^{-3}	2.5595	80	5.2150×10^{-4}	3.3749
159	7.1671×10^{-3}	1.3187	160	1.4000×10^{-4}	3.7250
319	2.6865×10^{-3}	2.6678	320	3.6260×10^{-5}	3.8610
			640	9.1910×10^{-6}	3.9452

Table 10: Interface derivative errors for example #3

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