

# A DYNAMICAL SYSTEMS APPROACH TO TRAVELING WAVE SOLUTIONS FOR LIQUID/VAPOR PHASE TRANSITION

HAITAO FAN AND XIAO-BIAO LIN

*Dedicated to Professor George Sell's 70th birthday*

ABSTRACT. We study the existence of liquefaction and evaporation waves by the methods derived from dynamical systems theory. A traveling wave solution is a heteroclinic orbit with the wave speed as a parameter. We give sufficient and necessary conditions for the existence of such heteroclinic orbit. After analyzing the local unstable and stable manifolds of two equilibrium points, we show that there exists at least one orbit connecting the local unstable manifold of one equilibrium point to the local stable manifold of another equilibrium point. The method is known as the shooting method in the literature.

## 1. INTRODUCTION

Dynamic flows involving liquid/vapor phase transition is an important phenomenon occurring in many engineering processes. For retrograde fluids, i.e. fluids with high specific heat capacities, such flows can be approximated by assuming that the temperature is constant.

The one-dimensional case of the system describing such flows in Lagrangian coordinates is

$$(1.1) \quad \begin{aligned} v_t - u_x &= 0, \\ u_t + p(\lambda, v)_x &= \epsilon u_{xx}, \\ \lambda_t &= \frac{1}{\gamma} w(\lambda, v) + \beta \lambda_{xx}, \end{aligned}$$

where  $v$  is the specific volume,  $u$  the velocity of the fluid,  $\lambda$  the weight portion of vapor in the liquid/vapor mixture,  $\epsilon$  the viscosity,  $\beta$  the diffusion coefficient and  $\gamma > 0$  the typical reaction time. The pressure  $p(\lambda, v)$  in (1.1) is a smooth function that satisfies

$$(1.2) \quad p_v < 0 < p_\lambda, \quad p_{vv} > 0.$$

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Although  $p_{vv} > 0$  was assumed in many previous papers, we do not use it in the proof of the existence of liquefaction and evaporation waves (Theorem 2.4 and Theorem 2.6).

Figure 1.1 shows the graph of a typical pressure function, where  $p_e$  is the equilibrium pressure at which liquid and vapor can coexist and  $m, M$  are the Maxwell points.

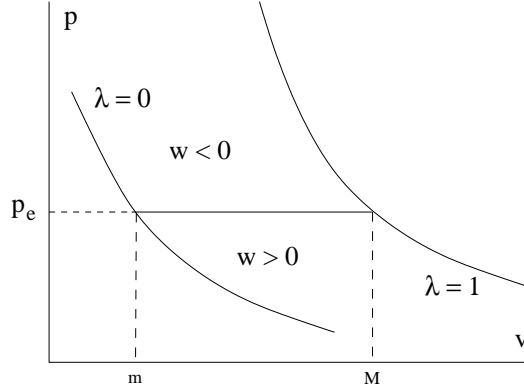


FIGURE 1.1. The pressure function  $p = p(\lambda, v)$  for some fixed  $\lambda$ .

The function  $w(\lambda, v)$  represents the rate of vapor initiation and growth:

$$(1.3) \quad w(v, \lambda) = (p(\lambda, v) - p_e)\lambda(\lambda - 1).$$

A traveling wave of (1.1) is a solution of the form  $(u, v, \lambda)(s)$ , where  $s = \frac{x-ct}{\epsilon}$  and  $c$  is the speed of the wave. With  $u' = du/ds$ , we have

$$(1.4) \quad \begin{aligned} -cv' - u' &= 0, \\ -cu' + p' &= u'', \\ -c\lambda' &= aw(\lambda, v) + b\lambda'', \\ (u, v, \lambda)(\pm\infty) &= (u_{\pm}, v_{\pm}, \lambda_{\pm}), \end{aligned}$$

where  $a = \epsilon/\gamma$ ,  $b = \beta/\epsilon$ . Because  $\lambda$  is the weight portion of the vapor in the liquid/vapor mixture, we only admit solutions with  $0 \leq \lambda \leq 1$ .

From the third equation of (1.4), equilibrium points  $(\lambda_{\pm}, v_{\pm})$  must satisfy  $w(\lambda, v) = 0$ , which has three branches of solutions:  $\lambda = 0$  or  $\lambda = 1$  or  $p(\lambda, v) = p_e$ .

From the first two equations of (1.4),  $c^2v' + p' = u'' = -cv''$ . Integrating from  $-\infty$  to  $s$  we have:

$$-cdv/ds = c^2(v - v_-) + (p - p_-).$$

Let  $s = b\xi$  and  $u' = du/d\xi$ . We have

$$(1.5) \quad \begin{aligned} \lambda'' &= -c\lambda' - abw(\lambda, v), \\ -\frac{c}{b}v' &= c^2(v - v_-) + (p(\lambda, v) - p_-). \end{aligned}$$

If the traveling wave connects  $E_{\pm}$  with  $(\lambda, v) = (\lambda_{\pm}, v_{\pm})$ , then

$$(1.6) \quad c^2(v_+ - v_-) + p(v_+, \lambda_+) - p(\lambda_-, v_-) = 0.$$

**Definition 1.1.** A liquefaction wave is a solution of (1.4) with

$$\begin{aligned} \lambda_- &= 0, \quad 0 < \lambda_+ \leq 1, \\ p(\lambda_{\pm}, v_{\pm}) &\geq p_e, \quad c^2 + p_v(\lambda_{\pm}, v_{\pm}) < 0, \end{aligned}$$

while an evaporation wave is that with

$$\begin{aligned} \lambda_- &= 1, \quad 0 \leq \lambda_+ < 1, \\ p(\lambda_{\pm}, v_{\pm}) &\leq p_e, \quad c^2 + p_v(\lambda_{\pm}, v_{\pm}) < 0. \end{aligned}$$

A collapsing wave is a solution of (1.4) with

$$\begin{aligned} 0 \leq \lambda_- &< 1, \quad \lambda_+ = 1, \\ p(\lambda_{\pm}, v_{\pm}) &\geq p_e, \quad c^2 + p_v(\lambda_{\pm}, v_{\pm}) > 0, \end{aligned}$$

while an explosion wave is that with

$$\begin{aligned} 0 < \lambda_- &\leq 1, \quad \lambda_+ = 0, \\ p(\lambda_{\pm}, v_{\pm}) &\leq p_e, \quad c^2 + p_v(\lambda_{\pm}, v_{\pm}) > 0. \end{aligned}$$

The definitions are summarized in the following table:

	$p \geq p_e$	$p \leq p_e$
subsonic $c^2 + p_v < 0$	liquefaction	evaporation
supersonic $c^2 + p_v > 0$	collapsing	explosion

Shearer (1982, 1983, 1986), Fan (1983) and Slemrod (1983) studied the Liquid/vapor phase transition through the  $p$ -system of conservation laws of hyperbolic-elliptic mixed type. Fan [4, 5, 6] proved the existence of liquefaction and evaporation waves. He also studied the stability of a simplified system consisting of a system of two conservation laws and a KPP equation.

The following methods were used in [4, 6] in proving the existence of traveling waves:

- (1) The Leray-Schauder degree theory.
- (2) The theory of monotone systems of parabolic PDEs.

A common feature to methods (1) and (2) is the adding of a small diffusion term  $\eta v_{xx}$  to the system,

$$\begin{aligned} v_t &= \eta v_{xx} + c^2(v - v_-) + p - p_-, \\ \lambda_t &= b\lambda_{xx} + a(p(\lambda, v) - p_e)\lambda(\lambda - 1). \end{aligned}$$

First one finds traveling waves for the system with small  $\eta > 0$ . Then one shows that there exists a sequence  $\eta_n \rightarrow 0$  such that the corresponding solutions  $(\lambda^{\eta_n}, v^{\eta_n}) \rightarrow (\lambda, v)$ . The limit is a traveling wave corresponding to  $\eta = 0$ .

We briefly describe the use of the Leray-Schauder degree theorem to our system. Consider the modified system:

$$\begin{aligned} \eta v'' + cv' &= -\theta(c^2(v - v_-) + p - p_-), \\ b\lambda'' + c\lambda' &= -\theta a w(v, \lambda), \quad -L < \xi < M, \\ (v, \lambda)(-L) &= (v_-, \lambda_-), \quad (v, \lambda)(M) = (\bar{v}_+, \bar{\lambda}_+). \end{aligned}$$

By choosing  $(\bar{v}_+, \bar{\lambda}_+)$  properly, one can show that there is a strict monotone solution for all  $\theta \in [0, 1]$ .

Write the system as an integral equation  $T(x, \theta) = x$ . Then  $T : \bar{\Omega} \times [0, 1] \rightarrow X$  is a compact operator in a real normed space. Moreover the solution exists if  $\theta = 0$ . Leray-Schauder degree asserts:

- (i) Assume that  $T(x, \theta) \neq x$  for  $x \in \partial\Omega$ ,  $\theta \in [0, 1]$ .
- (ii) The Leray-Schauder degree  $D_I(T(\cdot, 0) - I, \Omega) \neq 0$ .

Then for any  $0 \leq \theta \leq 1$ ,  $T(x, \theta) = x$  has at least one solution in  $\Omega$ .

We then find convergent subsequences of monotone solutions such that

- (i)  $L(n) \rightarrow \infty, M(n) \rightarrow \infty$ ,
- (ii)  $\bar{v}_n \rightarrow v_+, \bar{\lambda}_n \rightarrow \lambda_+$ ,
- (iii)  $\eta_n \rightarrow 0$ .

The limit of solutions is the traveling wave solution to the system with  $\eta = 0$ .

Next, we briefly describe the use of the ‘‘Method for Monotone Systems of PDES’’ to our system. Consider

$$\begin{aligned} v_t &= \eta v_{xx} + c^2(v - v_-) + p - p_-, \\ \lambda_t &= b\lambda_{xx} + a(p - p_e)\lambda(\lambda - 1). \end{aligned}$$

We can rewrite the system as

$$U_t = AU_{xx} + F(U, c),$$

where

$$F(U, c) = \begin{pmatrix} c^2(v - v_-) + p - p_- \\ a(p - p_e)\lambda(\lambda - 1) \end{pmatrix},$$

$$\nabla F = \begin{pmatrix} c^2 + p_v & p_\lambda \\ aP_v\lambda(\lambda - 1) & a(p - p_e)(2\lambda - 1) + aP_\lambda\lambda(\lambda - 1) \end{pmatrix}.$$

Under the sufficient conditions, we can verify that

- (1) the system is monotone: off-diagonal terms of  $\nabla F$  are positive.
- (2) The eigenvalues of  $\nabla F$  at  $U_-$  are negative,
- (3) The wave speed  $c$  is sufficiently large.
- (4) Other conditions for a monotone system are satisfied.

Then there exists a monotone solution  $U$  for small  $\eta$ . By letting  $\eta \rightarrow 0$  we find the limit of solutions which corresponds to solutions for the system with  $\eta = 0$ .

Using a geometric/dynamical system's method (shooting method), Fan and Lin simplified the proof of the existence of evaporation and liquefaction waves obtained in [4, 6]. We also rigorously proved the existence of collapsing and explosion waves that were only verified numerically before. Define  $h(\lambda, v) := c^2(v - v_-) + p(\lambda, v) - p_-$ . Then

$$\mathcal{C} := \{(\lambda, v) : h(\lambda, v) = 0\}$$

is the isocline for  $v$  due to (1.5). Based on (1.6),

$$h(\lambda, v) := c^2(v - v_\pm) + p(\lambda, v) - p_\pm.$$

The results obtained by the shooting method are summarized as follows:

**Theorem 1.1.** *Fan and Lin (2005-2008)*

(1) *The sufficient and necessary conditions for the existence of collapsing waves are:*

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_\pm, v_\pm) \geq 0.$$

(2) *The sufficient conditions for the existence of explosion waves are:*

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_-, v_-) \geq 0.$$

*The necessary conditions for the existence of explosion waves are:*

$$c^2 \geq 4ab|p(\lambda_+, v_+) - p_e|, \quad c^2 + p_v(\lambda_+, v_+) \geq 0.$$

(3) *If  $\lambda_+ = 0, 1$ , then the sufficient conditions for the existence of liquefaction or evaporation waves are:*

$$\text{For } v_- \leq v \leq v_+, c^2 + p_v(\lambda, v) < 0, \text{ if } \lambda = 0, 1,$$

and

$$c^2 \geq 4ab|p(\lambda_-, v_-) - p_e|.$$

(4) If  $p_+ = p_e, 0 < \lambda_+ < 1$ , then the sufficient conditions for the existence of liquefaction or evaporation waves are:

$$\begin{aligned} c^2 &\geq 4ab|p(\lambda_-, v_-) - p_e|, \\ c^2 + p_v(\lambda, v) &< 0, \text{ if } \lambda = 0 \leq \lambda \leq \lambda_+, v_- \leq v \leq v_+, \end{aligned}$$

and along the isocline  $\mathcal{C}$  for  $v$ ,

$$\sup_{\lambda} \left\{ \frac{abp_{\lambda}(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} \right\} < 1.$$

As proofs of the existence of collapsing and explosion waves were presented in a separate paper [8], in the rest of this paper we will study the existence of liquefaction and evaporation waves only. The existence of liquefaction waves for  $\lambda_- = 0, \lambda_+ = 1$  will be proved in Theorem 2.4 while the existence of liquefaction waves for  $\lambda_- = 0, 0 < \lambda_+ < 1, p_+ = p_e$  will be proved in Theorem 2.6. Similar proofs apply to the evaporation waves and will not be presented in this paper.

For liquefaction and evaporation waves, it will be shown in Lemma 2.1 that the wave speed  $c$  is positive. Therefore, from (1.6),

$$(1.7) \quad c = \sqrt{-\frac{p(\lambda_+, v_+) - p(\lambda_-, v_-)}{v_+ - v_-}}.$$

In [4, 6], Fan proved that liquefaction and evaporation waves exist if the wave speed  $c > 0$  satisfies  $c \geq 2\sqrt{ab|p(\lambda_-, v_-) - p_e|}$ . On the other hand, if the speeds satisfy  $c \leq 2\sqrt{ab|p(\lambda_+, v_+) - p_e|}$ , then there is no liquefaction or evaporation waves.

The locations of  $(v_{\pm}, \lambda_{\pm})$  for both waves are depicted in Figure 1.2.

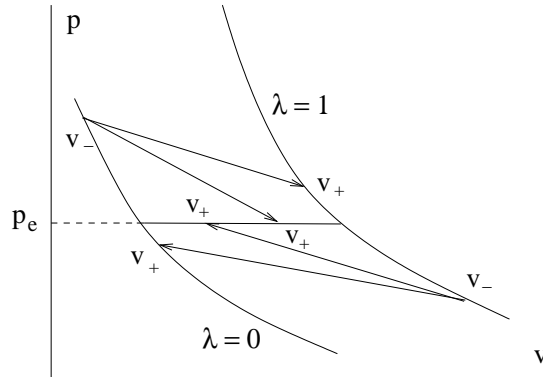


FIGURE 1.2. The points  $(v_{\pm}, \lambda_{\pm})$  of liquefaction waves ( $p_{\pm} \geq p_e$ ) and evaporation waves ( $p_{\pm} \leq p_e$ ). The arrows point to the fronts of the waves.

Recall that the pressure  $p(\lambda, v)$  satisfies

$$p_v < 0 < p_\lambda,$$

and the growth rate  $w$  is

$$w(\lambda, v) = (p - p_e)\lambda(\lambda - 1).$$

Since  $p_\lambda > 0$ , the function  $p = p(\lambda, v)$  can be solved for  $\lambda = \lambda^*(v, p)$ . For each  $v = v_0$ , with  $v_m < v_0 < v_M$ ,  $w = w(\lambda, v_0)$  has three zeros:  $\lambda = 0$ ,  $\lambda_e(v_0) = \lambda^*(v_0, p_e)$ ,  $\lambda = 1$ . However, for  $v_0 < v_m$  or  $v_0 > v_M$ ,  $w = w(\lambda, v_0)$  has two zeros:  $\lambda = 0$ ,  $\lambda = 1$ .

Let  $\mathcal{P} \subset \mathbb{R}^3$  be an open set bounded by finitely many smooth surfaces.

**Definition 1.2.** For each  $P \in \mathcal{P}$  such that the  $\Phi(\xi_0, P) \in \partial\mathcal{P}$  for some  $\xi_0 > 0$ , there exists the first touch time  $\xi_1$  such that

$$\Phi(\xi_1, P) \in \partial\mathcal{P}, \text{ while } \Phi(\xi, P) \in \mathcal{P} \text{ for } 0 \leq \xi < \xi_1.$$

If the first touch time for  $P$  exists, then define the first touch point as  $B(P) := \Phi(\xi_1, P)$ .

**Lemma 1.2.** Assume that there exist two mutually disjoint open subsets of  $\partial\mathcal{P}$ :  $S_1$  and  $S_2$  such that,

(1) for any  $P \in S_j$ ,  $j = 1, 2$ , there exists a small  $\epsilon > 0$  such that  $\Phi(\xi, P) \in \mathcal{P}$  for  $-\epsilon < \xi < 0$ . Moreover, the flow  $\Phi(\xi, \cdot)$  is transverse to  $S_1$  or  $S_2$ .

(2) For any  $P \in \mathcal{P}$  such that  $\Phi(\xi, P) \in \partial\mathcal{P}$  for some  $\xi > 0$ , we have  $B(P) \in S_1 \cup S_2$ .

(3) There exists a smooth curve segment  $\overline{P_1 P_2}$  in  $\mathcal{P}$  such that  $B(P_1) \in S_1$ ,  $B(P_2) \in S_2$ .

Under these conditions, there exists a  $P_0 \in \overline{P_1 P_2}$  such that  $\Phi(\xi, P_0)$  remains in  $\mathcal{P}$  for all  $\xi > 0$ .

## 2. EXISTENCE OF LIQUEFACTION AND EVAPORATION WAVES

In this section we present the proof of the existence of liquefaction waves for the case  $\{\lambda_- = 0\} \rightarrow \{\lambda_+ = 1\}$ , or the case  $\{\lambda_- = 0\} \rightarrow \{p_+ = p_e\}$ . The same proof applies to the evaporation waves with some minor changes.

System (1.5) can be written as a first order system of three variables  $(\lambda, \mu, v)$ :

$$(2.1) \quad \begin{aligned} \lambda' &= \mu, \\ \mu' &= -c\mu - abw(\lambda, v), \\ -\frac{c}{b}v' &= c^2(v - v_-) + (p(\lambda, v) - p_-). \end{aligned}$$

We look for a heteroclinic solution of (2.1) connecting the equilibrium points  $E_{\pm} := \{(\lambda_{\pm}, \mu_{\pm}, v_{\pm})\}$ .

Equilibrium states are the zeros of the right hand side of (2.1).

$$(2.2) \quad \begin{aligned} \mu &= 0, \\ w(\lambda, v) &= 0, \\ c^2(v - v_-) + p(\lambda, v) - p_- &= 0. \end{aligned}$$

The solutions of  $w = 0$  form three branches:  $\lambda = 0$ ,  $1$  and  $p(\lambda, v) = p_e$ . The graph of (2.2) with a given  $c$  is a straight line in the  $(v, p)$  plane, see Figure 1.2. However, the graph of (2.2) in the  $(\lambda, v)$  plane is the isocline  $\mathcal{C}$  for  $v$ .

For each  $v$  with  $v_m < v < v_M$ , by solving  $p(\lambda, v) = p_e$  for  $\lambda$ , we have

$$\lambda = \lambda_e(v).$$

The equilibrium  $E_-$  is on  $\lambda = 0$  with  $p_- > p_e$  or on  $\lambda = 1$  with  $p_- < p_e$ .

The equilibrium  $E_+$  is on the line  $\lambda = 1, p_+ > p_e$  or on  $p = p_e, 0 < \lambda < 1$  (liquefaction wave). The equilibrium  $E_+$  is on  $\lambda = 0, p < p_e$  or on  $p = p_e, 0 < \lambda < 1$  (evaporation wave).

Let  $p_+ = p(\lambda_+, v_+)$ . The wave speed  $c$  and  $v_{\pm}$  are now related by (1.6):  $c^2(v_+ - v_-) + (p_+ - p_-) = 0$ .

From (1.6), for the liquefaction wave  $p_+ < p_-$ , we must have  $v_+ > v_-$ ; while for the evaporation wave  $p_+ > p_-$ , we must have  $v_+ < v_-$ .

**2.1. Eigenvalues and eigenvectors at equilibrium points.** In this subsection, we first show that if  $c > 0$ , then the equilibrium  $E_-$  corresponding to  $\lambda = 0, 1$  is a saddle with exactly two positive eigenvalues and one negative eigenvalue, while the equilibrium  $E_+$  has one positive eigenvalues and two negative eigenvalues. The traveling wave solution we look for is a heteroclinic solution connecting saddle to saddle. Moreover, as  $\xi \rightarrow \pm\infty$ , the orbit of the traveling wave starts at the two dimensional local unstable manifold  $W_{loc}^u(E_-)$  and ends at the two dimensional local stable manifold  $W_{loc}^s(E_+)$ .

The linear variational system at equilibrium point is

$$\begin{pmatrix} \Lambda \\ M \\ V \end{pmatrix}' = A \begin{pmatrix} \Lambda \\ M \\ V \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 0 & 1 & 0 \\ -abw_{\lambda} & -c & -abw_v \\ -\frac{b}{c}p_{\lambda} & 0 & -bc - \frac{b}{c}p_v \end{pmatrix}.$$

Eigenvalues  $r$  are determined by

$$\det(rI - A) = \begin{vmatrix} r & -1 & 0 \\ abw_{\lambda} & r + c & abw_v \\ \frac{b}{c}p_{\lambda} & 0 & r + \frac{b}{c}(p_v + c^2) \end{vmatrix} = 0.$$

We first study eigenvalues at the equilibrium  $E_{\pm}$  with  $\lambda = 0$  or  $1$ ,

$$\begin{aligned} w_{\lambda}(\lambda, v) &= (2\lambda - 1)(p - p_e), \quad w_v(\lambda, v) = 0, \\ \det(rI - A) &= (r^2 + cr + w_{\lambda})(r + \frac{b}{c}(p_v + c^2)). \end{aligned}$$

Eigenvalues at  $\lambda = 0, 1$  are

$$\begin{aligned} r_1 &= -c/2 - \sqrt{(c/2)^2 - ab(2\lambda - 1)(p(\lambda, v) - p_e)}, \\ r_2 &= -c/2 + \sqrt{(c/2)^2 - ab(2\lambda - 1)(p(\lambda, v) - p_e)}, \\ r_3 &= -\frac{b}{c}(p_v(\lambda, v) + c^2). \end{aligned}$$

**Lemma 2.1.** (1) Assume that  $c > 0$ . At  $E_-$ , assume that  $c^2 + p_v(\lambda, v) < 0$  and  $(2\lambda - 1)(p - p_e) < 0$ . Then the equilibrium  $E_-$  has two positive eigenvalues and one negative eigenvalue. At  $E_+$ , assume that  $c^2 \geq 4ab|p(\lambda, v) - p_e|$  and  $c^2 + p_v(\lambda, v) < 0$ . Then if  $E_+$  is on the line  $\lambda = 0, 1$ , it has two negative eigenvalues and one positive eigenvalue.

(2) Assume that  $c < 0$ . At  $E_-$ , assume that  $c^2 + p_v(\lambda, v) < 0$  and  $(2\lambda - 1)(p - p_e) < 0$ . Then the equilibrium  $E_-$  has two negative eigenvalues and one positive eigenvalue. At  $E_+$ , assume that  $c^2 \geq 4ab|p(\lambda, v) - p_e|$  and  $c^2 + p_v(\lambda, v) < 0$ . Then if  $E_+$  is on the line  $\lambda = 0, 1$ , it has two positive eigenvalues and one negative eigenvalue.

*Proof.* Proof of (1): We always have  $r_3 > 0$  for  $\lambda = 0$  or  $1$ . If  $\lambda = 0$ ,  $p > p_e$  or if  $\lambda = 1$ ,  $p < p_e$ , we have

$$ab(2\lambda - 1)(p(\lambda, v) - p_e) < 0,$$

and hence  $r_1 < 0$  and  $r_2 > 0$ . Thus  $E_-$  has two unstable eigenvalues and one stable eigenvalue. If  $\lambda = 1$ ,  $p > p_e$  or if  $\lambda = 0$ ,  $p < p_e$ , we use  $c^2 \geq 4ab|p(\lambda, v) - p_e|$  to show  $r_1, r_2$  are real and  $r_1, r_2 < 0$ . Thus  $E_+$  has two stable eigenvalues and one unstable eigenvalue.

The proof of (2) is completely similar and shall be omitted.  $\square$

Since in the case (2), a heteroclinic connection from  $E_-$  to  $E_+$  usually does not happen, we shall assume  $c > 0$ .

**2.2. Existence of liquefaction waves for  $\lambda_- = 0$ ,  $\lambda_+ = 1$ .** In this subsection, we consider the liquefaction wave connecting  $\lambda_- = 0$  to  $\lambda_+ = 1$ . The liquefaction wave that connects  $\lambda_- = 0$  to  $p_+ = p_e$  shall be constructed later. A new assumption used in this subsection is:

(H1) For  $v_- \leq v \leq v_+$ ,  $c^2 + p_v(\lambda, v) < 0$ , if  $\lambda = 0, 1$ .

The traveling waves satisfy the following system of equations:

$$(2.3) \quad \lambda' = \mu, \quad \mu' = -c\mu - abw(\lambda, v),$$

$$(2.4) \quad \frac{-c}{b}v' = c^2(v - v_-) + (p(\lambda, v) - p_-).$$

As from Lemma 2.1, we assume that  $c > 0$ .

The isocline for  $v$  means  $\mathcal{C} := \{(\lambda, v) : v' = 0\}$ . Clearly  $(\lambda, v) \in \mathcal{C}$  iff  $h(\lambda, v) = 0$ . It is easy to see that on the two equilibrium points,  $(\lambda_{\pm}, v_{\pm}) \in \mathcal{C}$ .

Due to (H1), on the line  $\lambda_- = 0$  we have  $c^2 + p_v < 0$ . If  $v > v_-$ , then  $h(0, v) < 0$ . Therefore  $v' > 0$  if  $v_- < v < v_+$  and  $\lambda = 0$ . Similarly, due to (H1) again, if  $\lambda = 1$ , we can show that  $v' < 0$  if  $v_- < v < v_+$ . Now for each  $v \in (v_-, v_+)$ , there exists a unique  $\lambda \in (0, \lambda_+)$  such that  $h(\lambda, v) = 0$ , denoted by

$$\lambda = \lambda_c(v).$$

Due to the fact  $p_\lambda > 0$ ,  $\lambda_c(v)$  is a smooth function of  $v \in (v_-, v_+)$ .

In general  $\lambda_c(v)$  may not be a monotone function as depicted in Figure 2.1.

The isocline for  $v$  divides the rectangle  $(\lambda_-, \lambda_+) \times (v_-, v_+)$  into two parts. Let

$$\mathcal{N} := \{(\lambda, v) : v_- < v < v_+, 0 < \lambda < \lambda_c(v)\}.$$

If  $(\lambda, v) \in \mathcal{N}$ , then  $v'(\xi) > 0$ . Let  $EF$  be the curve on which  $p = p_e$  and  $v_- < v < v_+, \lambda_- < \lambda < \lambda_+$ , see Figure 2.1. Then  $EF \subset \mathcal{N}$  where  $v' > 0$ . This can be shown as follows. Since on  $EF$ ,  $p = p_e < p_+$  and  $v < v_+$ , we have

$$h(\lambda, v) = c^2(v - v_+) + (p - p_+) < 0.$$

Consider a pentahedron shaped solid  $\mathcal{P}$  in  $(\lambda, \mu, v)$  space bounded by the five surfaces:

$$\text{Left side } \mathcal{F}_\ell := \{\lambda = 0\};$$

$$\text{Back side } \mathcal{F}_k := \{v = v_+, 0 < \lambda < 1, 0 < \mu < -c(\lambda - 1)/2\};$$

$$\text{Front side } \mathcal{F}_f := \{v_- < v < v_+, \lambda = \lambda_c(v), 0 < \mu < -c(\lambda - 1)/2\};$$

$$\text{Slant side } \mathcal{F}_s := \{c(\lambda - 1)/2 + \mu = 0, 0 < \lambda < \lambda_c(v), v_- < v < v_+\};$$

$$\text{Bottom side } \mathcal{F}_b := \{\mu = 0, 0 < \lambda < 1\}.$$

The bottom side is further divided into  $\mathcal{F}_b = \mathcal{F}_{b1} \cup \mathcal{F}_{b2}$ , with

$$\mathcal{F}_{b1} := \mathcal{F}_b \cap \{p(\lambda, v) \geq p_e\},$$

$$\mathcal{F}_{b2} := \mathcal{F}_b \cap \{p(\lambda, v) < p_e\}.$$

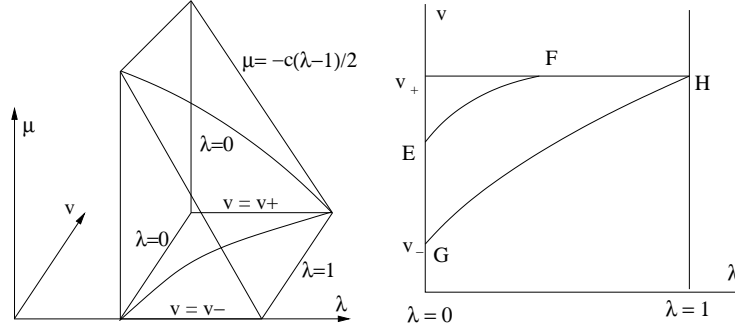


FIGURE 2.1. The pentahedron and the top view of its base  $\mathcal{F}_b$ . On the curve  $EF$ ,  $p(\lambda, v) = p_e$ . On the isocline  $GH$ ,  $v' = 0$ .

For each interior point  $P$  of  $\mathcal{P}$ , let  $B(P)$  be the first touch point on  $\partial\mathcal{P}$  as in Definition 1.2.

(1) Since  $d\lambda/d\xi = \mu > 0$  inside  $\mathcal{P}$ ,  $B(P) \notin \mathcal{F}_\ell$ .

(2) On  $\mathcal{F}_k$ , we have  $v' = -\frac{b}{c}(p(v_+, \lambda) - p_+)$ . Since  $p_\lambda > 0$  and  $\lambda < 1$ , we have  $p(v_+, \lambda) < p(v_+, 1) = p_+$ . Thus,  $v' > 0$ . It is possible  $B(P) \in \mathcal{F}_k$ .

(3) On  $\mathcal{F}_f$ , we have  $\lambda' > 0$  and  $v' = 0$ . Let the outward normal of  $\mathcal{F}_f$  be  $\mathbf{n} = \{(\lambda, \mu, v) = (1, 0, -d\lambda_c(v)/dv)\}$ , and let the vector field be  $\mathbf{f}$ . Then  $\mathbf{n} \cdot \mathbf{f} > 0$ . The flow starts on  $\mathcal{F}_f$  must leave  $\mathcal{P}$  trasversely. It is possible that  $B(P) \in \mathcal{F}_f$  for some  $P \in \mathcal{P}$ .

(4) On the interior of  $\mathcal{F}_{b2}$  we have  $d\mu/d\xi < 0$  due to  $w > 0$  for  $0 < \lambda < 1$  and  $p < p_e$ . It is possible that  $B(P) \in \mathcal{F}_{b2}$ .

(5) On the interior of  $\mathcal{F}_{b1}$ , we have  $d\mu/d\xi = -abw(\lambda, v) > 0$  due to  $w < 0$  for  $0 < \lambda < 1$  and  $p > p_e$ . Thus  $B(P)$  is not in the interior of  $\mathcal{F}_{b1}$ .

If  $B(P) \in \{p = p_e\} \cap \mathcal{F}_{b1}$  at the first touch time  $\xi_1 > 0$ , then from (2.3) it is easy to verify that  $\mu(\xi_1) = \mu'(\xi_1) = 0$  and  $\mu''(\xi_1) < 0$ . Therefore, there exists  $\delta > 0$  such that  $\mu(\xi) < 0$  if  $\xi_1 - \delta \leq \xi < \xi_1$ , contradicting to  $\xi_1$  being the first touch time. So  $B(P)$  cannot be on the line  $\{p = p_e\} \cap \mathcal{F}_{b1}$ .

The following lemma shows that  $B(P) \notin \mathcal{F}_s$ .

**Lemma 2.2.** *The first touch point with the boundary  $B(P)$  is not on the slant side  $\mathcal{F}_s$ .*

*Proof.* The inward normal of the slant side  $\mathcal{F}_s := \{c(\lambda - 1)/2 + \mu = 0\}$  is

$$\mathbf{n} = (\mathbf{n}_\lambda, \mathbf{n}_\mu, \mathbf{n}_v) = (-c/2, -1, 0).$$

The vector field is

$$\mathbf{f} = (\mathbf{f}_\lambda, \mathbf{f}_\mu, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on  $\mathcal{F}_s$ ,

$$\mathbf{n} \cdot \mathbf{f} = -c\mu/2 + c\mu + abw(\lambda, v) > 0.$$

Using  $\mu = -c(\lambda - 1)/2$ , we have

$$(2.5) \quad \mathbf{n} \cdot \mathbf{f} = (1 - \lambda)((c/2)^2 - ab\lambda(p - p_e)).$$

Since  $(\lambda, v)$  satisfies

$$c^2(v - v_-) + (p - p_-) \leq 0, \quad \text{and } v \geq v_-,$$

we have  $p \leq p_-$ . Therefore,

$$(c/2)^2 > ab(p_- - p_e) \geq ab\lambda(p(\lambda, v) - p_e).$$

It follows that  $\mathbf{n} \cdot \mathbf{f} > 0$ , see (2.5). Therefore  $B(P) \notin \mathcal{F}_s$ .  $\square$

Let us check the edges of  $\mathcal{P}$  (not including  $E_+$ ). The four edges that lie on  $\lambda = 0$  cannot contain  $B(P)$  as shown by (1).

Among the other four edges, two of them bound  $\mathcal{F}_s$ , so they cannot contain  $B(P)$  due to  $\mathbf{n} \cdot \mathbf{f} > 0$  as in Lemma 2.2. What left are the two more edges that bound  $\mathcal{F}_{b1}$  (not including  $\{p = p_e\}$ ). They cannot contain  $B(P)$  due to  $\mu' > 0$ .

We have shown that if  $B(P)$  is the first time that  $\Phi(\xi, P)$  hits of the boundary of  $\mathcal{P}$ , either it lies on  $S_1 := \mathcal{F}_f$  or it lies on  $S_2 := \mathcal{F}_{b2} \cup \mathcal{F}_k \cup \{\mu = 0, v = v_+, 0 < \lambda < \lambda_e(v_+)\}$ .

Moreover,  $B(P)$  cannot belong to the four boundaries of  $\mathcal{F}_\ell$ , the three boundaries of  $\mathcal{F}_s$  and the three boundaries of  $\mathcal{F}_f$ . The point  $B(P)$  can belong to the common boundary of  $\mathcal{F}_k$  and  $\mathcal{F}_{b2}$  but not the common boundaries of  $\mathcal{F}_k$  and  $\mathcal{F}_{b1}$ .

**Lemma 2.3.** *There exist  $P_1, P_2 \in W_{loc}^u(E_-) \cap \mathcal{P}$  such that  $B(P_1) \in S_1$  and  $B(P_2) \in S_2$ .*

*Proof.* To start the shooting method, we list facts about  $W^u(E_-)$ : Let  $r_1 < 0 < r_2$  be the two eigenvalues for (2.3). Let  $r_3 > 0$  be the eigenvalue with the eigenvector  $(0, 0, 1)$ .

(1) The line  $\{\lambda = 0\}$  is on  $W^u(E_-)$ . On this line, we have

$$v' > 0 \text{ if } v > v_-, \quad v' < 0 \text{ if } v < v_-.$$

(2)  $W_{loc}^u(E_-)$  is two dimensional with two linearly independent tangent vectors

$$(\Lambda, M, V) = (0, 0, 1) \text{ and } (\Lambda, M, V) = (1, r_2, 0).$$

Based on this, we can express the local unstable manifold as

$$W_{loc}^u(E_-) = \{(\lambda, v, \mu) : -\epsilon_1 < \lambda < \epsilon_1, v_- - \epsilon_2 < v < v_- + \epsilon_2, \mu = \mu^*(\lambda, v)\}.$$

Moreover, if  $\lambda > 0$ , then  $\mu^* > 0$ .

Recall that the isocline  $\mathcal{C}$  for  $v$  can be expressed as  $\lambda = \lambda_c(v)$  with  $\lambda_c(v_-) = 0$ , and  $d\lambda_c(v_-)/dv > 0$ . Choose  $\bar{v}$  with  $v_- < \bar{v} < v_- + \epsilon_2$  so that  $0 < \lambda_c(\bar{v}) < \epsilon_1$ .

For each  $0 < \lambda_1 < \lambda_c(\bar{v})$ , define a line segment  $\overline{P_1 P_2}$  on  $W_{loc}^u(E_-)$ :

$$\overline{P_1 P_2} := \{(\lambda, v, \mu) : \lambda_1 \leq \lambda \leq \lambda_c(\bar{v}), v = \bar{v}, \mu = \mu^*(\lambda, v)\}.$$

It is parametrized by  $\lambda$  with  $\lambda = \lambda_c(\bar{v})$  corresponding to  $P_1$  and  $\lambda = \lambda_1$  corresponding to  $P_2$ . It is also clear that  $\overline{P_1 P_2}$  is in  $\mathcal{P}$  except for the point  $P_1$ .

Since the flow on the  $v$ -axis is transversal to the plane  $\{v = v^+\}$ , assuming that  $\lambda_1$  is sufficiently small so that  $P_2$  is close to the  $v$ -axis on  $W^u(E_-)$ , then the orbit  $\Phi(\xi, P_2)$  stays close to the  $v$ -axis until it hits  $v = v_+$  transversely. It is easy to show that  $B(P_2)$  will hit the boundary of  $\mathcal{P}$  in  $S_2$ . On the other hand,  $P_1$  is on  $S_1$  and the flow  $\Phi(\xi, P_1)$  leaves  $\mathcal{P}$  transversely at  $P_1 \in S_1$ .  $\square$

**Theorem 2.4.** *Consider the liquefaction waves with  $\lambda_- = 0$ ,  $\lambda_+ = 1$ . Assume  $c^2 + p_v(\lambda, v) < 0$  if  $\lambda = 0, 1$  and  $v \in [v_-, v_+]$  and  $c^2 > 4ab|p(\lambda_-, v_-) - p_e|$ . Then there exists a liquefaction wave connecting  $E_-$  to  $E_+$ . The  $(\lambda, v)$  components of the wave are monotone.*

*Proof.* There exists a relatively open subset  $\mathcal{O}_1$  of  $\overline{P_1 P_2}$  containing every  $P$  such that  $B(P) \in S_1$ . There exists also a relatively open subset  $\mathcal{O}_2$  of  $\overline{P_1 P_2}$  containing every  $P$  such that  $B(P) \in S_2$ . It is also clear that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are mutually disjoint. Since  $P_1 \in \mathcal{O}_1$ ,  $P_2 \in \mathcal{O}_2$  and  $\overline{P_1 P_2}$  is a connected set,

$$(2.6) \quad \overline{P_1 P_2} - (\mathcal{O}_1 \cup \mathcal{O}_2)$$

is nonempty. Let  $P$  be a point from (2.6). Then  $\Phi(\xi, P)$  cannot hit the boundary of  $\mathcal{P}$ . It must stay inside  $\mathcal{P}$  and approach the equilibrium  $E_+$ . Also  $\Phi(\xi, P) \rightarrow E_-$  since  $P \in W_{loc}^u(E_-)$ .  $\square$

**2.3. Existence of liquefaction waves for  $\lambda_- = 0$ ,  $0 < \lambda_+ < 1$ ,  $p_+ = p_e$ .** Assume  $c > 0$  as before so that the equilibrium  $E_-$  corresponding to  $\lambda = 0, 1$  is a saddle with exactly two positive eigenvalues and one negative eigenvalue. We do not know exactly what are the eigenvalues for  $E_+$  when  $p_+ = p_e, \lambda \neq 0, 1$ . However, it is not used in the proof of Theorem 2.6.

Assume that

$$c^2 + p_v(\lambda, v) < 0$$

is satisfied throughout the region  $\lambda \in [0, \lambda_e], v \in [v_-, v_+]$ . The isocline  $\mathcal{C} := \{v' = 0\}$  can be solved from the equation

$$c^2(v - v_+) + p(\lambda, v) - p_+ = 0,$$

and the solution can be expressed as

$$v = v_c(\lambda), \quad 0 \leq \lambda \leq \lambda_+,$$

$$\frac{dv_c(\lambda)}{d\lambda} = \frac{-p_\lambda}{c^2 + p_v} > 0.$$

We look for the liquefaction wave connecting  $\lambda_- = 0$  to  $p_+ = p_e$ . The traveling wave satisfies (2.3) and (2.4). As from Lemma 2.1, we assume that  $c > 0$ .

Consider a pentahedron shaped solid  $\mathcal{P}$  in  $(\lambda, \mu, v)$  space bounded by the five surfaces:

$$\text{Left side } \mathcal{F}_\ell := \{\lambda = 0\};$$

$$\text{Back side } \mathcal{F}_k := \{v = v_+, 0 < \lambda < \lambda_e, 0 < \mu < -c(\lambda - \lambda_e)/2\};$$

$$\text{Front side } \mathcal{F}_f := \{0 < \lambda < \lambda_e, v = v_c(\lambda), 0 < \mu < -c(\lambda - \lambda_e)/2\};$$

$$\text{Slant side } \mathcal{F}_s := \{c(\lambda - \lambda_e)/2 + \mu = 0, 0 < \lambda < \lambda_e, v_c(\lambda) < v < v_+\};$$

$$\text{Bottom side } \mathcal{F}_b := \{\mu = 0, 0 < \lambda < \lambda_e\}.$$

The bottom side is further divided into  $\mathcal{F}_b = \mathcal{F}_{b1} \cup \mathcal{F}_{b2}$ , with

$$\mathcal{F}_{b1} := \mathcal{F}_b \cap \{p(\lambda, v) \geq p_e\},$$

$$\mathcal{F}_{b2} := \mathcal{F}_b \cap \{p(\lambda, v) < p_e\}.$$

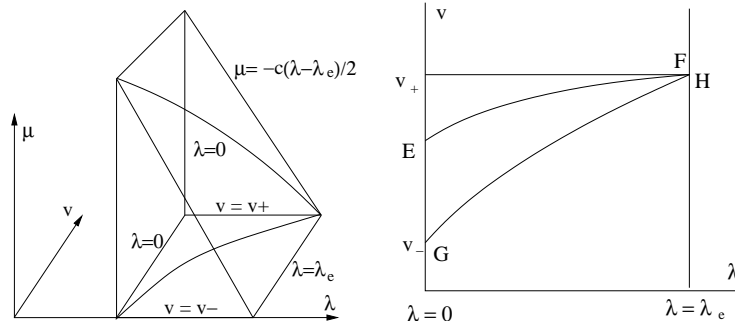


FIGURE 2.2. The pentahedron and the top view of its base  $\mathcal{F}_b$ . On the curve  $EF$ ,  $p(\lambda, v) = p_e$ . On the isocline  $GH$ ,  $v' = 0$ .

Let  $P$  be an interior point of  $\mathcal{P}$  and  $B(P)$  be the first touch point of the orbit at  $\partial\mathcal{P}$ . Just as in §2.2, we can show:

- (1)  $B(P) \notin \mathcal{F}_\ell$ .
- (2) It is possible that  $B(P) \in \mathcal{F}_k$ .
- (3) It is possible that  $B(P) \in \mathcal{F}_f$ .
- (4) It is possible that  $B(P) \in \mathcal{F}_{b_2}$ .
- (5)  $B(P)$  is not in the interior of  $\mathcal{F}_{b_1}$ . Also  $B(P)$  cannot be on the line  $\{p = p_e\} \cap \mathcal{F}_{b_1}$ .

The following lemma shows that  $B(P) \notin \mathcal{F}_s$ .

**Lemma 2.5.** *Assume that along the isocline  $\mathcal{C}$ , we have*

$$\sup_{\lambda} \left\{ \frac{abp_{\lambda}(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} \right\} < 1.$$

*Then  $B(P)$  is not on the slant side  $\mathcal{F}_s$ .*

*Proof.* The inward normal of the slant side  $\mathcal{F}_s := \{c(\lambda - \lambda_e)/2 + \mu = 0\}$  is

$$\mathbf{n} = (\mathbf{n}_{\lambda}, \mathbf{n}_{\mu}, \mathbf{n}_v) = (-c/2, -1, 0).$$

The vector fields are

$$\mathbf{f} = (\mathbf{f}_{\lambda}, \mathbf{f}_{\mu}, \mathbf{f}_v) = (\mu, -c\mu - abw(\lambda, v), v').$$

We want to show that on  $\mathcal{F}_s$ ,

$$\mathbf{n} \cdot \mathbf{f} = -c\mu/2 + c\mu + abw(\lambda, v) > 0.$$

Using  $\mu = -c(\lambda - \lambda_e)/2$ , we have

$$(2.7) \quad \mathbf{n} \cdot \mathbf{f} = (\lambda_e - \lambda) \left( (c/2)^2 - ab \frac{w(\lambda, v)}{\lambda - \lambda_e} \right).$$

Recall that  $p_v < 0$ , thus  $\partial w / \partial v > 0$ . Since on  $\mathcal{F}_s$ , we have  $v_c(\lambda) < v < v_+$ , therefore

$$(2.8) \quad \frac{w(\lambda, v)}{\lambda - \lambda_e} < \frac{w(\lambda, v_c(\lambda))}{\lambda - \lambda_e} \leq \frac{1}{4} \left| \frac{p(\lambda, v_c(\lambda)) - p(\lambda_e, v_c(\lambda_e))}{\lambda - \lambda_e} \right|,$$

by the fact  $\lambda(1 - \lambda) < 1/4$  and  $p(\lambda_e, v_c(\lambda_e)) = 0$ .

The difference quotient can be estimated by

$$\begin{aligned} \sup_{\lambda} \left| \frac{dp(\lambda, v_c(\lambda))}{d\lambda} \right| &= \sup_{\lambda} |p_{\lambda} + p_v(dv_c(\lambda)/d\lambda)| \\ &= \sup_{\lambda} \left| p_{\lambda} + p_v \frac{-p_{\lambda}}{c^2 + p_v} \right| = \sup_{\lambda} \left| \frac{c^2 p_{\lambda}}{c^2 + p_v} \right|. \end{aligned}$$

If the assumption of the lemma is satisfied, then from (2.7) and (2.8), we have  $\mathbf{n} \cdot \mathbf{f} > 0$ . □

**Theorem 2.6.** *Consider the liquefaction waves with  $\lambda_- = 0$ ,  $0 < \lambda_+ < 1$ ,  $p_+ = p_e$ . Assume that  $c^2 + p_v(\lambda, v) < 0$  throughout the region and*

$$\sup_{\lambda} \left\{ \frac{abp_{\lambda}(\lambda, v_c(\lambda))}{|c^2 + p_v(\lambda, v_c(\lambda))|} < 1, \right.$$

*along the isocline  $\mathcal{C}$  for  $v$ . Then there exists a liquefaction wave connecting  $E_-$  to  $E_+$  with  $p_+ = p_e$ . The  $(\lambda, v)$  components of the wave are monotone.*

*Proof.* The rest of the proof of the existence of the liquefaction waves follows exactly like the case where  $\lambda_+ = 1$ .  $\square$

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DEPARTMENT OF MATHEMATICS, GEORGETOWN UNIVERSITY, WASHINGTON, DC 20057

*E-mail address:* fan@math.georgetown.edu

NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695-8205

*E-mail address:* xblin@math.ncsu.edu