Solutions to last HW

2. Take $\underset{\sim}{X_n}=\begin{pmatrix}1 & Y_1\\ 1 & Y_2\\ \vdots & \vdots\\ 1 & Y_{n-1}\end{pmatrix}$, $\underset{\sim}{\beta}=\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$, $\underset{\sim}{\mathcal{E}_n}=\begin{pmatrix}e_2\\ \vdots\\ e_n\end{pmatrix}$, $\underset{\sim}{Y_n}=\begin{pmatrix}Y_2\\ \vdots\\ Y_n\end{pmatrix}$

Then our model is

$$\underset{\sim}{Y_n}=\underset{\sim}{X_n}^T\beta+\underset{\sim}{\mathcal{E}_n}$$

The least square estimate for β is

$$\hat\beta=(\underset{\sim}{X_n}^T\underset{\sim}{X_n})^{-1}\underset{\sim}{X_n}^T\underset{\sim}{Y_n}=(\underset{\sim}{X_n}^T\underset{\sim}{X_n})^{-1}\underset{\sim}{X_n}^T(\underset{\sim}{X_n}^T\beta+\underset{\sim}{\mathcal{E}_n})$$

$$=\beta+(X_n^TX_n)^{-1}\underset{\sim}{X_n}^T\underset{\sim}{\mathcal{E}_n}$$

$$\therefore\begin{pmatrix}\hat\alpha_0-\alpha_0\\ \hat\alpha_1-\alpha_1\end{pmatrix}=\begin{pmatrix}n-1 & \sum_{t=2}^n Y_{t-1}\\ \sum_{t=2}^n Y_{t-1} & \sum_{t=2}^n Y_{t-1}^2\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=2}^n e_t\\ \sum_{t=2}^n Y_{t-1}e_t\end{pmatrix}$$

① $\frac{\sqrt{n-1}}{\sqrt{n}}\frac{1}{\sqrt{n-1}}\sum_{t=2}^n e_t\xrightarrow{\mathcal{D}}N(0,\sigma^2)$

② $\sum_2^n Y_{t-1}e_t=\sum_2^n[(t-1)\alpha_0+Y_0+X_{t-1}]e_t$

$$=\alpha_0\sum_2^n(t-1)e_t+Y_0\sum_2^n e_t+\sum_2^n X_{t-1}e_t$$

We know $\sum_2^n e_t=O_p(\sqrt n)$, $\sum_2^n X_t e_t=O_p(n)$

and $E\left(\sum_2^n(t-1)e_t\right)=0$,

$$Var\left(\sum_2^n(t-1)e_t\right)=\sigma^2\sum_2^n(t-1)^2=\sigma^2\frac{(n-1)n(2n-1)}{6}$$

$$\therefore\frac{1}{n^{\frac32}}\sum_2^n(t-1)e_t\xrightarrow{\mathcal{D}}N\left(0,\frac{\sigma^2}{3}\right)$$

$$\Rightarrow\frac{1}{n^{\frac32}}\sum_2^n Y_{t-1}e_t=\frac{\alpha_0}{n^{\frac32}}\sum_2^n(t-1)e_t+O_p\left(\frac1n\right)+O_p\left(\frac{1}{\sqrt n}\right)$$

$$\xrightarrow{\mathcal{D}}N\left(0,\frac{\alpha_0^2\sigma^2}{3}\right)$$

③ $cov\left(\frac{1}{\sqrt n}\sum_{t=2}^n e_t,\ \frac{1}{n^{\frac32}}\sum_2^n Y_{t-1}e_t\right)=\frac{1}{n^2}\sum_2^n cov(e_t,Y_{t-1}e_t)$

$$=\frac{1}{n^2}\sum_2^n E(Y_{t-1})E(e_t^2)=\frac{\sigma^2}{n^2}\sum_2^n E[(t-1)\alpha_0+Y_0+X_{t-1}]$$

$$= \frac{\sigma^2}{n^2} \sum_2^n \{(t-1)\alpha_0 + E(y_0)\} \doteq \frac{\alpha_0 \sigma^2}{n^2} \cdot \frac{n(n-1)}{2} + \frac{\sigma^2}{n^2} n E(y_0)$$

$$\longrightarrow \frac{\alpha_0}{2}\sigma^2$$

$$\therefore \begin{pmatrix} \frac{1}{\sqrt{n}} \sum e_t \\ \frac{1}{n^{\frac{3}{2}}} \sum y_{t-1} e_t \end{pmatrix} \xrightarrow{\mathcal{D}} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0^2}{3} \end{pmatrix} \sigma^2 \right)$$

Now, $\frac{n-1}{n} \longrightarrow 1$

$$\frac{1}{n^2} \sum_2^n y_{t-1} = \frac{1}{n^2} \sum_2^n \{(t-1)\alpha_0 + y_0 + X_{t-1}\}$$

$$= \frac{\alpha_0}{n^2} \frac{n(n-1)}{2} + \frac{(n-1)y_0}{n^2} + \frac{\sum X_{t-1}}{n^2}$$

$$\xrightarrow{P} \frac{\alpha_0}{2}$$

$$\frac{1}{n^3} \sum_2^n y_{t-1}^2 = \frac{1}{n^3} \sum_2^n \{(t-1)\alpha_0 + y_0 + X_{t-1}\}^2$$

$$= \frac{\alpha_0^2}{n^3} \sum_2^n (t-1)^2 + \frac{n y_0^2}{n^3} + \frac{\sum_2^n X_{t-1}^2}{n^3} + \frac{2\alpha_0 y_0 \sum (t-1)}{n^3} + \frac{2\alpha_0 \sum (t-1) X_{t-1}}{n^3}$$

$$\xrightarrow{P} \frac{\alpha_0^2}{3}$$

$$\therefore \begin{pmatrix} \frac{n-1}{n} & \frac{1}{n^2}\sum y_{t-1} \\ \frac{1}{n^2}\sum y_{t-1} & \frac{1}{n^3}\sum y_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{\sqrt{n}} \sum e_t \\ \frac{1}{n^{\frac{3}{2}}} \sum y_{t-1} e_t \end{pmatrix}$$

$$= \left[\begin{pmatrix} \frac{1}{\sqrt{n}} & \\ & \frac{1}{n^{\frac{3}{2}}} \end{pmatrix} \begin{pmatrix} n-1 & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{n}} & \\ & \frac{1}{n^{\frac{3}{2}}} \end{pmatrix} \right]^{-1} \begin{bmatrix} \frac{1}{\sqrt{n}} & \\ & \frac{1}{n^{\frac{3}{2}}} \end{bmatrix} \begin{pmatrix} \sum e_t \\ \sum y_{t-1} e_t \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{n} & 0 \\ 0 & n^{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} n-1 & \sum y_{t-1} \\ \sum y_{t-1} & \sum y_{t-1}^2 \end{pmatrix} \begin{pmatrix} \sum e_t \\ \sum y_{t-1} e_t \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{n} & 0 \\ 0 & n^{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \hat{\alpha}_0 - \alpha_0 \\ \hat{\alpha}_1 - \alpha_1 \end{pmatrix} \xrightarrow{\mathcal{D}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0^2}{3} \end{pmatrix}^{-1} \begin{pmatrix} 1 & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0^2}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0^2}{3} \end{pmatrix}^{-1} \right)$$

$$= N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \frac{\alpha_0^2}{3} \end{pmatrix}^{-1} \right)$$

3. (i) $$\hat{\alpha}_{1,w} = \frac{\sum_2^n Y_{t-1}Y_t}{\sum_2^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2} = \frac{\sum_2^n Y_{t-1}Y_t}{\sum_2^n Y_{t-1}^2} \times \frac{\sum_2^n Y_{t-1}^2}{\sum_2^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2}$$

$$= \hat{\alpha}_{1,OLS} + \hat{\alpha}_{1,OLS}\left(\frac{\sum_2^n Y_{t-1}^2}{\sum_2^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2} - 1\right)$$

$$= \hat{\alpha}_{1,OLS} + \hat{\alpha}_{1,OLS} \times \frac{Y_1^2 - \frac{1}{n}\sum_1^n Y_t^2}{\sum_2^{n-1} Y_{t-1}^2 + \frac{1}{n}\sum_1^n Y_t^2}$$

Since $\sum_1^n Y_t^2 = O_p(n)$,

$$\frac{\frac{1}{n}Y_1^2 - \frac{1}{n^2}\sum Y_t^2}{\frac{1}{n}\sum Y_{t-1}^2 + \frac{1}{n^2}\sum Y_t^2} \xrightarrow{P} 0$$

$\therefore \hat{\alpha}_{1,w} \xrightarrow{\mathcal{D}} \hat{\alpha}_{1,OLS}$

$\therefore n[\hat{\alpha}_{1,w} - \alpha_1] \xrightarrow{\mathcal{D}} N(0, 1-\alpha_1^2)$

(ii) When $\alpha_1 = 1$,

$Y_t = Y_0 + X_t$ where $X_t = \sum_{i=1}^t e_i$

$$\hat{\alpha}_{1,w} = \frac{\sum_2^n Y_{t-1}Y_t}{\sum_2^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2} = \frac{\sum_2^n Y_{t-1}Y_t}{\sum_2^{n-1} Y_t^2 + Y_1^2} \times \frac{\sum_2^{n-1} Y_t^2 + Y_1^2}{\sum_2^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2}$$

$$= \hat{\alpha}_{1,OLS} \times \frac{\sum_1^{n-1} Y_t^2}{\sum_1^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2}$$

$\because \frac{1}{n^2}\sum_1^{n-1} Y_t^2 = \frac{1}{n^2}\sum_1^{n-1}(Y_0 + X_t)^2 = \frac{1}{n^2}\sum_1^{n-1} Y_0^2 + \frac{2}{n^2}Y_0\sum_1^{n-1} X_t + \frac{1}{n^2}\sum_1^{n-1} X_t^2$

We know $\frac{X_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \sigma\int_0^1 w(x)dx$

$$\frac{1}{n^2}\sum X_{t-1}^2 \xrightarrow{\mathcal{D}} \sigma^2\int_0^1 w^2(x)\,dx$$

$\therefore \frac{1}{n^2}\sum_1^{n-1} Y_0^2 = \frac{(n-1)Y_0^2}{n^2} \xrightarrow{P} 0$

$\frac{2}{n^2}Y_0\sum_1^{n-1} X_t = \frac{2}{\sqrt{n}}Y_0\frac{\overline{X}_n}{\sqrt{n}} \xrightarrow{P} 0$

$\Rightarrow \frac{1}{n^2}\sum_1^{n-1} Y_t^2 \xrightarrow{\mathcal{D}} \sigma^2\int_0^1 w^2(x)dx$

$$\therefore \sum_1^{n-1} Y_t^2 = O_p(n^2) \Rightarrow \frac{1}{n^3}\sum_1^n Y_t^2 \xrightarrow{P} 0$$

$$\therefore \frac{\sum_1^{n-1} Y_t^2}{\sum_1^{n-1} Y_t^2 + \frac{1}{n}\sum_1^n Y_t^2} = \frac{\frac{1}{n^2}\sum_1^{n-1} Y_t^2}{\frac{1}{n^2}\sum_1^{n-1} Y_t^2 + \frac{1}{n^3}\sum_1^n Y_t^2} \xrightarrow{P} 1$$

$$\hat{\alpha}_{1,W} \xrightarrow{\mathcal{D}} \hat{\alpha}_{1,OLS}$$

$$\therefore n[\hat{\alpha}_{1,W} - 1] \xrightarrow{\mathcal{D}} \frac{\frac{1}{2}[W^2(1) - 1]}{\int_0^1 W^2(x)\,dx}$$ Dickey–Fuller distribution

4. $$\hat{\alpha}_1 = \frac{\sum_5^n Y_{t-4}Y_t}{\sum_5^n Y_{t-4}^2} = \frac{\sum_{t=5}^n Y_{t-4}(\alpha_1 Y_{t-4} + e_t)}{\sum_5^n Y_{t-4}^2} = \alpha_1 + \frac{\sum Y_{t-4} e_t}{\sum Y_{t-4}^2}$$

(i) When $|\alpha_1| < 1$

Since $Y_{t-4} e_t = Z_t$ are iid

$E(Z_t) = E(Y_{t-4})E(e_t) = 0$

$Var(Z_t) = E(Z_t^2) = E(Y_{t-4}^2)E(e_t^2) = Var(Y_{t-4})Var(e_t) = \frac{\sigma^4}{1-\alpha_1^2}$

$$\frac{1}{\sqrt{n-4}}\sum_{t=5}^n Y_{t-4} e_t \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma^4}{1-\alpha_1^2}\right)$$

And $\frac{1}{n-4}\sum_{t=5}^n Y_{t-4}^2 \xrightarrow{P} \gamma_Y(0) = \frac{\sigma^2}{1-\alpha_1^2}$

$$\therefore \sqrt{n}[\hat{\alpha}_1 - \alpha_1] = \frac{\sqrt{n}}{\sqrt{n-4}} \frac{\frac{1}{\sqrt{n-4}}\sum_5^n Y_{t-4}Y_t}{\frac{1}{n-4}\sum_5^n Y_{t-4}^2} \xrightarrow{\mathcal{D}} N\left(0, \frac{\frac{\sigma^4}{1-\alpha_1^2}}{\left(\frac{\sigma^2}{1-\alpha_1^2}\right)^2}\right)$$

$$\equiv N(0, 1-\alpha_1^2)$$

(ii) When $\alpha_1 = 1$

$$Y_t = Y_{t-4} + e_t = Y_{t-8} + e_{t-4} + e_t = \cdots$$

Let $X_t^{(0)} = \sum_{k=1}^{[\frac{t}{4}]} e_{4k}$, $X_t^{(1)} = \sum_{k=1}^{[\frac{t}{4}]} e_{4k-1}$, $X_t^{(2)} = \sum_{k=1}^{[\frac{t}{4}]} e_{4k-2}$

$X_t^{(3)} = \sum_{k=1}^{[\frac{t}{4}]} e_{4k-3}$

When $t=4J$ where $2\le J\le T$, $Y_t = Y_0 + e_4 + e_8 + \cdots + e_t = Y_0 + X_t^{(0)}$

$t=4J-1 \qquad Y_t = Y_{-1} + e_3 + e_7 + \cdots + e_t = Y_{-1} + X_t^{(1)}$

$t=4J-2 \quad \cdots \quad Y_t = Y_{-2} + e_2 + e_6 + \cdots + e_t = Y_{-2} + X_t^{(2)}$

$t=4J-3 \qquad Y_t = Y_{-3} + e_1 + e_5 + \cdots + e_t = Y_{-3} + X_t^{(3)}$

$$\sum_{t=5}^{n} Y_{t-4}e_t = \sum_{k=1}^{T-1} Y_{4k-3}e_{4k+1} + \sum_{k=1}^{T-1} Y_{4k-2}e_{4k+2} + \sum_{k=1}^{T-1} Y_{4k-1}e_{4k+3} + \sum_{k=1}^{T-1} Y_{4k}e_{4k+4}$$

$$= \sum_{k=1}^{T-1}(Y_{-3}+X_t^{(3)})e_{4k+1} + \cdots + \sum_{k=1}^{T-1}(Y_0 + X_t^{(0)})e_{4k+4}$$

$$= \sum_{k=1}^{T-1}(e_1+e_5+\cdots+e_{4k-3})e_{4k+1} + \sum_{k=1}^{T-1}(e_2+e_5+\cdots+e_{4k-2})e_{4k+2}$$

$$+ \sum_{k=1}^{T-1}(e_3+e_7+\cdots+e_{4k-1})e_{4k+3} + \sum_{k=1}^{T-1}(e_4+e_8+\cdots+e_{4k})e_{4k+4}$$

$$+ Y_{-3}\sum_{k=1}^{T-1} e_{4k+1} + \cdots + Y_0\sum_{k=1}^{T-1} e_{4k+4}$$

$$\therefore \frac{1}{n}\sum_5^n Y_{t-4}e_t = \frac{1}{n}\,\frac{(\sum_{k=1}^{T} e_{4k-3})^2 - \sum_{k=1}^{T} e_{4k-3}^2}{2} + \frac{1}{n}\,\frac{(\sum_{k=1}^{T} e_{4k-2})^2 - \sum_{k=1}^{T} e_{4k-2}^2}{2}$$

$$+\frac{1}{n}\,\frac{(\sum_{k=1}^{T} e_{4k-1})^2 - \sum_{k=1}^{T} e_{4k-1}^2}{2} + \frac{1}{n}\,\frac{(\sum_{k=1}^{T} e_{4k})^2 - \sum_{k=1}^{T} e_{4k}^2}{2} + O_p\left(\frac{1}{\sqrt{n}}\right)$$

$$= \frac{1}{4}\,\frac{(\frac{1}{\sqrt{T}}\sum_{k=1}^{T} e_{4k-3})^2 - \frac{1}{T}\sum_{k=1}^{T} e_{4k-3}^2}{2} + \cdots + \frac{1}{4}\,\frac{(\frac{1}{\sqrt{T}}\sum_{k=1}^{T} e_{4k})^2 - \frac{1}{T}\sum_{1}^{T} e_{4k}^2}{2}$$

$$+O_p\left(\frac{1}{\sqrt{n}}\right)$$

$$\xrightarrow{\mathcal{D}} \frac{\sigma^2}{2}(\chi_1^2 - 1)$$

$$\sum_5^n Y_{t-4}^2 = \sum_{k=1}^{T-1} Y_{4k-3}^2 + \sum_{k=1}^{T-1} Y_{4k-2}^2 + \sum_{k=1}^{T-1} Y_{4k-1}^2 + \sum_{k=1}^{T-1} Y_{4k}^2$$

$$= \sum_{k=1}^{T-1}(Y_{-3} + X_{4k-3}^{(3)})^2 + \cdots + \sum_{k=1}^{T-1}(Y_0 + X_{4k}^{(0)})^2$$

$$= (T-1)(Y_{-3}^2 + Y_{-2}^2 + Y_{-1}^2 + Y_0^2)$$

$$+ 2Y_{-3}\sum_1^{T-1} X_{4k-3}^{(3)} + 2Y_{-2}\sum_1^{T-1} X_{4k-2}^{(2)} + \cdots + 2Y_0\sum_1^{T-1} X_{4k}^{(0)}$$

$$+ \sum_1^{T-1}(X_{4k-3}^{(3)})^2 + \cdots + \sum_1^{T-1}(X_{4k}^{(0)})^2$$

Since $\sum_1^{T-1} X_{4k-3}^{(3)} = O_p(T^{\frac{3}{2}}), \ \cdots \ \sum_1^{T-1} X_{4k}^{(0)} = O_p(T^{\frac{3}{2}})$

$$\sum_{1}^{T-1} \left(X_{4k-3}^{(3)}\right)^2 = O_p(T^2) \quad \cdots \quad \sum_{1}^{T-1} \left(X_{4k}^{(0)}\right)^2 = O_p(T^2)$$

$$\therefore \frac{1}{n^2}\sum_{5}^{n} Y_{t-4}^2 = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{(4T)^2}\sum_{1}^{T-1}\left(X_{4k-3}^{(3)}\right)^2 + \cdots + \frac{1}{(4T)^2}\sum_{1}^{T-1}\left(X_{4k}^{(0)}\right)^2$$

$$\xrightarrow{\mathcal{D}} \frac{1}{4}\sigma^2 \int_0^1 W^2(x)\,dx$$

$$\therefore n[\hat{\alpha}_1 - 1] = \frac{\frac{1}{n}\sum Y_{t-4} e_t}{\frac{1}{n^2}\sum Y_{t-4}^2} \xrightarrow{\mathcal{D}} \frac{\frac{\sigma^2}{2}(W^2(1) - 1)}{\frac{1}{4}\sigma^2 \int_0^1 W^2(x)\,dx}$$

$$= \frac{2(W^2(1) - 1)}{\int_0^1 W^2(x)\,dx}$$