MA 574
Mathematical and Experimental Modeling of Physical Processes II

PROJECT 1: ACOUSTIC WAVES IN A PVC PIPE DUE DATE: MARCH 1, 2001

The objective of this project is to study two types of boundary conditions for acoustic waves propagation in a PVC pipe. Experiments with two different boundary conditions (hardwall and foam) will be carried out in the laboratory using a harmonic oscillator at the other boundary condition. These data are then used to obtain reflection coefficients over a wide range of frequencies. The reflection coefficients, in turn, are used to estimate unknown parameters in the models for boundary conditions.

The acoustic wave motion in a fluid is described by either the acoustic pressure, p, or the velocity potential, ϕ. These two quantities are related by $p(t,x)=\rho\phi_t$ and satisfy the wave equation

$$\phi_{tt}=c^2\phi_{xx}, \quad 0<x<l \tag{1}$$

where c is the speed of sound, l is the length of the pipe, and the motion is assumed to be one dimensional. Two types of boundary conditions will be considered:

Oscillating boundaries. The interaction of the boundary at $x=l$ and the interior pressure is modeled by a damped harmonic oscillator and is described by

$$m\delta_{tt}+d\delta_t+k\delta=-\rho\phi_t(t,l). \tag{2}$$

Here δ is the displacement (normal) of the boundary in the direction interior to the fluid. It is also assumed that the boundary surface is not penetrable by the fluid, that is,

$$\delta_t(t)=\phi_x(t,l). \tag{3}$$

We note that the solution to the wave equation (1) has the form

$$\phi(t,x)=F(t-x/c)+G(t+x/c) \tag{4}$$

where the first term on the right hand side of (4) describes a wave propagates to the right and the second term corresponds to a left propagating wave. From Eq. (3) and by integrating, we obtain

$$\delta(t)=-\frac{1}{c}(\tilde{F}(t)-\tilde{G}(t)) \tag{5}$$

where, without loss of generality, the constant of integration is set to zero and $\tilde{F}(t) = F(t - l/c),\ \tilde{G}(t) = G(t + l/c)$. Substituting (5) into (20) yields

$$m\tilde{G}'' + (d + \rho c)\tilde{G}' + k\tilde{G} = m\tilde{F}'' + (d - \rho c)\tilde{F}' + k\tilde{F} \quad (6)$$

Now, assume that the incident wave $\tilde{F}$ to the boundary at $x = l$ (which is generated by a harmonic input at $x = 0$), is a simple harmonic of frequency $\omega / 2\pi$. That is,

$$\tilde{F}(t) = A_0 e^{i\omega t}, \quad (7)$$

so that the right hand side of (6) is a harmonic forcing function. It follows that the *steady state* solution of (6) is also harmonic with the same frequency

$$\tilde{G}(t) = R(\omega) A_0 e^{i\omega t}, \quad (8)$$

where the complex coefficient, $R(\omega)$, is called the *reflection coefficient*. Substituting equations (7) and (8) into (6) we have a relation for the reflection coefficient for the oscillating boundary condition model given by

$$R(\omega) = \frac{m\omega^2 - i(d - \rho c)\omega - k}{m\omega^2 - i(d + \rho c)\omega - k}. \quad (9)$$

Damped elastic boundaries. The boundary conditions for this model, in terms of the acoustic pressure, have the form

$$\alpha p(t,l) + \beta p_t(t,l) + c p_x(t,l) = 0$$

Assuming harmonic incident wave as previously, we obtain the damped elastic reflection coefficient

$$R(\omega) = \frac{i\omega(1-\beta) - \alpha}{i\omega(1+\beta) + \alpha}. \quad (10)$$

This project involves the following steps:

1. The acoustic pressure anywhere in the pipe for planar wave propagation is given by the following equation:

$$p(t,x) = A(\omega) e^{i\omega(t - x/c)} + A(\omega) R(\omega) e^{i\omega(t + x/c)}$$

   By measuring the pressure, $p(t, x_j)$, at a number of axial locations, x_j, and for a specific angular frequency $\tilde{\omega}$, an inverse least squares problem can be formulated to estimate both complex coefficients, $A(\tilde{\omega})$ and $R(\tilde{\omega})$. Considering both physical hardwall and foam type of boundary conditions at $x = l$ and collecting two corresponding sets of experimental data, one can use these to estimate $R(\tilde{\omega})$. This data will be denoted by $R^d(\tilde{\omega})$, over the range of frequencies from 50 Hz to 500Hz.

2. In this problem, we will evaluate how well the oscillating boundary and damped elastic boundary models described by formulas (9) and (10) fit the experimental data $R^d(\tilde{\omega}_j)$. That is, we will determine the set of parameters, (m, d, k, ρ) and (α, β), so that the functional

$$\sum_{i=1}^{N} | R^d(\omega_i) - R(\omega_i) |^2$$

is minimized. Here, N is the number of measurements R^d at frequencies $f_i = \omega_i / 2\pi$. In your report, discuss which model (9) or (10), or both, is (or are) best to describe the hardwall and the foam type of boundary condition.