An Application of Stochastic Control Theory to Financial Economics*

Wendell H. Fleming†
Division of Applied Mathematics
Brown University
Providence, RI 02912

Tao Pang‡
Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205

Dec. 3, 2002
(Revised May 05, 2003)

Abstract

We consider a portfolio optimization problem which is formulated as a stochastic control problem. Risky asset prices obey a logarithmic Brownian motion, and interest rates vary according to an ergodic Markov diffusion process. The goal is to choose optimal investment and consumption policies to maximize the infinite horizon expected discounted HARA utility of consumption. A dynamic programming principle is used to derive the dynamic programming equation (DPE). The sub-supersolution method is used to obtain existence of solutions of the DPE. The solutions are then used to derive the optimal investment and consumption policies.

Keywords: Portfolio optimization, dynamic programming equations, subsolutions and supersolutions.

AMS Subject Classifications: 93E20, 60H30

Abbreviated Title: An Application of Stochastic Control Theory.

*This research is partially supported by NSF Grant DMS-9970852 through Brown University.
†Email: whf@cfm.brown.edu Fax: (401) 863-1355.
‡Email: tpang@math.ncsu.edu Fax: (919) 515-3798.

1 Introduction

In the classical Merton portfolio optimization problem, an investor dynamically allocates wealth between a risky and a riskless asset and chooses a consumption rate, with the goal of maximizing total expected discounted utility of consumption. For HARA utility function the Merton problem has a simple explicit solution. See for example Fleming and Soner [FlSo] Example 5.2. In the Merton model, the interest rate r of the riskless asset is a constant and the risky asset price fluctuates randomly according to a logarithmic Brownian motion. However, in our real world, even for the money in the bank, the interest rate may fluctuate from time to time. Therefore, in the present paper we assume that the "riskless" interest rate r_t is an ergodic Markov diffusion process on the real line $-\infty < r < \infty$. A typical example is the Vasicek model, in which r_t is of Ornstein - Uhlenbeck type. In addition, the change of interest rate could be correlated with the price fluctuating of the risky asset. A recent example is that, the US Federal Reserve has lowered the interest rate several times since 2000, due to the bad performance of the US stock markets. We also take this into account in this paper. Please see Section 2 for details.

Another motivation for our work comes from models for optimal investment, production and consumption, of a kind considered by Fleming and Stein [FlSt2]. This interpretation of our model will be explained at the end of Section 2. See also Fleming and Pang [FlP].

We use the dynamic programming method. The stochastic control problem which we consider has state variables x_t, r_t, where x_t is the wealth. The controls are the fraction u_t of wealth in the risky asset and $c_t = \frac{C_t}{x_t}$ where C_t is the consumption rate. The state dynamics are the stochastic differential equations (2.1) $-$ (2.4). For HARA utility, the value function $V(x,r)$ is a homogeneous function of x: $V(x,r) = \frac{1}{\gamma}x^\gamma W(r)$, where γ is the HARA parameter. For $\gamma > 0$, a source of technical difficulty is that $W(r)$ increases rapidly to infinity as $|r| \to \infty$. In fact, $Z(r) = \log W(r)$ should grow quadratically as $|r| \to \infty$. The dynamic programming equation (2.14) for $V(x,r)$ is equivalent to a nonlinear ordinary differential equation (2.22) for $Z(r)$. We call (2.22) the reduced dynamic programming equation.

We use a method of subsolution and supersolution to show that the reduced dynamic programming equation (2.22) has a solution $\tilde{Z}(r)$ with appropriate behavior as $|r| \to \infty$. The sub/supersolution method is developed in Section 3. It is applied in Section 4 with $\gamma > 0$, to find a classical solution $\tilde{Z}(r)$ to (2.22) which is bounded below and which grows at most quadratically as $|r| \to \infty$. A verification result (Theorem 3) then shows that $\tilde{Z}(r) = Z(r)$ and that the corresponding control policies $u^*(r), c^*(r)$ in formulas (4.60) are optimal. These results require that $0 < \gamma < \bar{\gamma}$ for suitable $\bar{\gamma} \le 1$. In Section 5 we consider $\gamma < 0$. In this case $\tilde{W}(r) = \exp(\tilde{Z}(r))$ decays to 0 as $|r| \to \infty$ like $|r|^{2(\gamma-1)}$. The verification result is Theorem 5 in this case.

The results in this paper are adapted from the second author's Ph.D thesis [Pang]. In Chapter 2 of [Pang], a related optimal investment problem on a finite time horizon $0 \le t \le T$ was also considered. The goal is then to choose an investment control u_t to maximize expected HARA utility of final wealth $E[\gamma^{-1}x_T^\gamma]$. This model is of a type previously considered by Bielecki and Pliska [BiPl], Zariphopoulou [Z], Fleming and Sheu [FlSh1]. The analysis for that finite horizon stochastic control problem is considerably simpler than for the optimal investment-consumption model considered in the present paper.

Fleming and Hernandez-Hernandez [FlHH] considered an investment/ consumption model in which the interest rate is constant but the volatility of the risky asset price is stochastic. The approach in [FlHH] has some features in common with the present paper. However, the methods and technical issues to be resolved in the two papers are different.

Our methods should apply to a wider class of stochastic control problems in which the dynamic programming equation reduces to an ODE of the form $-LZ = h(r, Z)$ as in (4.3). The function $h(r, Z)$ in (4.2) is the sum of a term $\gamma Q(r) - \beta$ and a decreasing function of Z. The function $Q(r)$ grows quadratically as $|r| \to \infty$. This feature significantly complicated the analyses in Section 4 and 5, in the cases $\gamma > 0$ and $\gamma < 0$.

2 The Dynamic Programming Equation

We use a logarithmic Brownian motion to describe the price P_t of the risky asset:

$$\frac{dP_t}{P_t} = bdt + \sigma_1 dw_{1,t},$$

where b, σ_1 are positive constants and $w_{1,t}$ is a standard 1-dimensional Brownian motion. Let x_t be the wealth at time t. The investment control u_t at time t is the fraction of wealth invested in the risky asset. So $(1 - u_t)$ is the fraction of the wealth invested on the riskless asset. Denote C_t the consumption rate at time t. For technical reasons, we take $c_t \equiv \frac{C_t}{x_t}$ as a control instead of C_t. Suppose the initial wealth is $x > 0$. Then the stochastic differential equation for the process x_t is

$$\begin{aligned} dx_t &= x_t[r_t + (b - r_t)u_t - c_t]dt + \sigma_1 u_t x_t dw_{1,t}, && (2.1)\\ x_0 &= x, && (2.2) \end{aligned}$$

where r_t is the interest rate of the riskless asset at time t. Instead of a constant interest rate in the classical Merton's model, we consider a randomly fluctuating interest rate model:

$$\begin{aligned} dr_t &= f(r_t)dt + \sigma_2 d\tilde{w}_t, && (2.3)\\ r_0 &= r, && (2.4) \end{aligned}$$

where σ_2 is a constant and $\tilde{w}_t$ is a standard 1-dimensional Brownian motion. In some cases, the fluctuation of the interest rate is correlated with the price change of the risky asset. To describe this, we let $w_t = (w_{1,t}, w_{2,t})'$ be a standard 2-dimensional Brownian motion. Define $\tilde{w}_t$ such that

$$d\tilde{w}_t = \rho dw_{1,t} + \sqrt{1-\rho^2}dw_{2,t}, \tag{2.5}$$

where $\rho \in [-1, 1]$ is a constant. Since $w_{1,t}$ and $w_{2,t}$ are independent, we have

$$E[dw_{1,t} \cdot d\tilde{w}_t] = \rho dt. \tag{2.6}$$

So ρ is the correlation coefficient.

In this paper, we will consider the generalized Vasicek model:

$$\begin{aligned} f(r) &\in \mathbf{C}^2(\mathbf{R}), && (2.7)\\ |f_{rr}(r)| &\le K(1 + |r|^\alpha), && (2.8)\\ -c_2 \le f_r(r) &\le -c_1, && (2.9) \end{aligned}$$

where K, α, c_1 and c_2 are positive constants.

We consider a HARA utility function $U(\cdot)$:

$$U(C) = \frac{1}{\gamma}C^\gamma, \ -\infty < \gamma < 1, \ \gamma \neq 0. \tag{2.10}$$

Our goal is then to maximize the objective function

$$J(x, r, u_\cdot, c_\cdot) \equiv E_{x,r}\int_0^\infty e^{-\beta t}U(c_t x_t)dt, \tag{2.11}$$

where $(u_\cdot, c_\cdot)$ belong to a class Π of admissible controls. Then our value function is

$$V(x, r) = \sup_{u_\cdot, c_\cdot} E_{x,r}\int_0^\infty e^{-\beta t}U(c_t x_t)dt. \tag{2.12}$$

We require that the control $(u_t, c_t; t \geq 0)$ is an $\mathbf{R}^2$-valued process. In addition, we require that it is $\mathcal{F}_t$-progressively measurable for some $(w_{1,t}, \tilde{w}_t)$-adapted increasing family of σ-algebras $(\mathcal{F}_t, t \geq 0)$. See Fleming and Soner [FlSo] Chapter 4 for details. In certain cases, (u_t, c_t) may be obtained from locally Lipschitz continuous control policies $(\underline{u}, \underline{c})$:

$$u_t = \underline{u}(t, x_t, r_t), \ c_t = \underline{c}(t, x_t, r_t),$$

where x_t is obtained by substituting these policies in (1.1).

We also assume that $c_t \geq 0$, and there is no constraint for the value of u_t. In other words, we take the u-value space $\mathbf{U} = (-\infty, \infty)$ in this paper. The negative value of u_t corresponds to disinvestment such as short-selling.

In addition, we require that

$$P\left(\int_0^T u_t^2 dt < \infty\right) = 1, \quad \forall T > 0. \tag{2.13}$$

Given this, we can use the Ito's differential rule to verify that

$$x_t = x \exp\left\{\int_0^t [r_s + (b - r_s)u_s - c_s - \frac{1}{2}\sigma_1^2 u_s^2] ds + \int_0^t \sigma_1 u_s dw_{1,s}\right\}$$

is a solution of (2.1) – (2.2). We can see that $x_t > 0$ as long as $x > 0$.

Remark 1 *The admissible control space Π will be specified later in Definition 3 ($\gamma > 0$ case) and Definition 4 ($\gamma < 0$ case). For fixed $\beta > 0$, there exists a constant $\bar{\gamma} \leq 1$ such that $0 < \gamma < \bar{\gamma}$ will insure that $V(x, r) < \infty$. For a constant interest r, a condition about β and γ is given in Fleming and Soner [FlSo] page 176.*

Remark 2 *The log utility case, which corresponds to HARA utility with $\gamma = 0$, is studied in Pang [Pang] Section 1.4. It is much easier to deal with.*

By the definition of $V(x, r)$, using the dynamic programming principle, we can obtain that the corresponding dynamic programming equation is

$$\begin{aligned} \beta V &= \sup_u \left[(b - r)uxV_x + \frac{1}{2}\sigma_1^2 u^2 x^2 V_{xx} + \rho\sigma_1\sigma_2 uxV_{xr}\right] + rxV_x \\ &\quad + f(r)V_r + \frac{1}{2}\sigma_2^2 V_{rr} + \sup_{c \geq 0}\left[-cxV_x + \frac{1}{\gamma}(cx)^\gamma\right]. \end{aligned} \tag{2.14}$$

For details, please refer to Fleming and Soner [FlSo] Section 4.5.

Since we consider a HARA utility function which is homogeneous in x with an order of γ, it is not hard to get the following lemma:

Lemma 1 *$V(x, r)$ is homogeneous in x with an order of γ.*

Proof. According to (2.1) – (2.2), for any $k > 0$, we have

$$\begin{aligned} dkx_t &= kx_t[r_t + (b - r_t)u_t - c_t]dt + \sigma_1 u_t kx_t dw_{1,t}, \\ kx_0 &= kx. \end{aligned}$$

Therefore,

$$\begin{aligned} J(kx, r, u., c.) &= E_{x,r}\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(c_t kx_t)^\gamma dt \\ &= k^\gamma E_{x,r}\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt \\ &= k^\gamma J(x, r, u., c.). \end{aligned}$$

Thus we have

$$\begin{aligned} V(x,r) &= \sup_{u.,c.} J(x,r,u.,c.) \\ &= \sup_{u.,c.} x^\gamma J(1,r,u.,c.) \\ &= x^\gamma V(1,r). \end{aligned}$$

That is, $V(x,r)$ is homogeneous in x. **Q.E.D.**

From Lemma 1, we can suppose that

$$V(x,r) = \frac{1}{\gamma} x^\gamma W(r). \tag{2.15}$$

Then, the differential equation for $W(r)$ can be written as

$$\begin{aligned} \frac{\beta}{\gamma} W &= \sup_u \left[(b-r)uW + \frac{1}{2}(\gamma-1)\sigma_1^2 u^2 W + \rho\sigma_1\sigma_2 u W_r \right] + rW \\ &\quad + \frac{1}{\gamma} f(r) W_r + \frac{1}{2\gamma}\sigma_2^2 W_{rr} + \sup_{c\geq 0} \left[-cW + \frac{1}{\gamma} c^\gamma \right]. \end{aligned}$$

By the definition of $V(x,r)$, it is not hard to know that the suitable $W(r)$ should be positive.

Actually, if $W(r) > 0$ and smooth enough, we can define

$$u^*(r) \equiv \frac{(b-r)W(r) + \rho\sigma_1\sigma_2 W_r(r)}{(1-\gamma)\sigma_1^2 W(r)}, \tag{2.16}$$

$$c^*(r) \equiv W(r)^{\frac{1}{\gamma-1}}. \tag{2.17}$$

Then we have

$$\begin{aligned} u^*(r) &\in \arg\max_u \left[(b-r)uW + \frac{1}{2}(\gamma-1)\sigma_1^2 u^2 W + \rho\sigma_1\sigma_2 u W_r \right], \\ c^*(r) &\in \arg\max_c \left[-cW + \frac{1}{\gamma} c^\gamma \right]. \end{aligned}$$

Actually, (u^*, c^*) will be verified to be the optimal control policy later in Section 4 and Section 5 for $\gamma > 0$ and $\gamma < 0$, respectively.

Now we can rewrite the differential equation of $W(r)$ as

$$\begin{aligned} &\frac{1}{2}\sigma_2^2 W_{rr} + \left[\frac{\gamma\rho\sigma_2(b-r)}{\sigma_1(1-\gamma)} + f(r) \right] W_r + \frac{\gamma\rho^2\sigma_2^2 W_r^2}{2(1-\gamma)W} \\ &\quad + [\gamma Q(r) - \beta] W + (1-\gamma) W^{\frac{\gamma}{\gamma-1}} = 0, \end{aligned} \tag{2.18}$$

where

$$Q(r) = \frac{(b-r)^2}{2(1-\gamma)\sigma_1^2} + r. \tag{2.19}$$

We can see that $Q(r)$ is quadratic with respect to r. Let

$$Z(r) \equiv \log W(r). \tag{2.20}$$

Then the ODE for $Z(r)$ is

$$\begin{aligned} &\frac{\sigma_2^2}{2} Z_{rr} + \frac{\sigma_2^2}{2}\left[1 + \frac{\gamma\rho^2}{1-\gamma} \right] Z_r^2 + \left[\frac{\gamma\rho\sigma_2(b-r)}{\sigma_1(1-\gamma)} + f(r) \right] Z_r \\ &\quad + \gamma Q(r) - \beta + (1-\gamma) e^{\frac{Z}{\gamma-1}} = 0. \end{aligned}$$

Define

$$H(r,z,p) \equiv -\frac{\sigma_2^2}{2}\left[1+\frac{\gamma\rho^2}{1-\gamma}\right]p^2 - \left[\frac{\gamma\rho\sigma_2(b-r)}{\sigma_1(1-\gamma)} + f(r)\right]p$$
$$-\gamma Q(r) + \beta - (1-\gamma)e^{\frac{z}{\gamma-1}}, \qquad (2.21)$$

then the equation for $Z(r)$ can be rewritten as

$$\frac{\sigma_2^2}{2}Z_{rr} = H(r,Z,Z_r). \qquad (2.22)$$

We call (2.22) the reduced DPE. Our goal is to find a suitable solution $\tilde{V}(x,r)$ of the DPE (2.14) and verify that $\tilde{V}(x,r)$ is equal to the value function defined by (2.12). To obtain $\tilde{V}(x,r)$, it is sufficient to find a suitable solution $Z(r)$ of (2.22). Then, $\tilde{V}(x,r) = \frac{1}{\gamma}x^\gamma e^{Z(r)}$ will be the desired solution of (2.14). Although (2.22) is a nonlinear equation, we can get some existence results by using a subsolution-supersolution method.

Investment, production and consumption model. In addition to Merton-type, small investor portfolio optimization problems with randomly fluctuation interest rates, another motivation for our work comes from considering models of the following kind. An economic unit has productive capital and also liabilities in the form of debt. Let K_t denote the worth of capital at time t and L_t the debt. K_t changes through investment, at rate I_t. Debt changes through interest payments, investment, consumption C_t and income from production Y_t:

$$dK_t = I_t dt \qquad (2.23)$$
$$dL_t = (r_t L_t + I_t + C_t - Y_t)dt. \qquad (2.24)$$

It is assume that productivity of capital fluctuates randomly about a mean rate b. This is expressed by writing (formally)

$$Y_t dt = K_t(bdt + \sigma_1 dw_{1,t}) \qquad (2.25)$$

with $w_{1,t}$ a Brownian motion as above. The constraints imposed are $K_t \geq 0, C_t \geq 0, x_t > 0$, where $x_t = K_t - L_t$ is the net worth of the economic unit. By subtracting (2.24) from (2.23) we find that x_t satisfies the stochastic differential equation (2.1) with

$$u_t = x_t^{-1}K_t, \quad c_t = x_t^{-1}C_t. \qquad (2.26)$$

If no bounds are imposed on the investment rate I_t, then u_t can be taken as the investment control and c_t the consumption control. The constraint $K_t \geq 0$ is equivalent to the "no short selling" constraint $u_t \geq 0$. We will ignore this constraint in the sections to follow. To include, it requires rather easy modifications. For example, in (2.14), the first sup would be taken over $u \geq 0$ rather than over all u.

In [FlSt2], a similar international finance and debt model was considered. In that interpretation the economic unit is a nation. Y_t represents the national gross domestic product and L_t is the foreign debt. However, instead of a "mean reverting" model (2.3) for the interest rate r_t, it is assumed in [FlSt2] that (formally)

$$r_t dt = rdt + \sigma_2 dw_{2,t}$$

with $w_{2,t}$ a Brownian motion. As in the Merton problem, there is an explicit solution in the model considered in [FlSt2]. However, if the interest rate r_t satisfies the SDE (2.3), then the optimal investment and consumption policies $u^*(r), c^*(r)$ depend on the solution $W(r)$ to a reduced dynamic programming equation as in (2.16) and (2.17). This differential equation, or the equivalent differential equation for $Z(r) = \log W(r)$ can be solved numerically.

3 Method of Subsolution and Supersolution

In this section, we will give an existence result for some type of ODEs which include (2.22). The method of subsolution and supersolution will be used. This idea is partially from [P], [BSW] and [W].

Consider a second order differential equation

$$Z_{rr} = \bar{H}(r, Z, Z_r). \tag{3.1}$$

First let us define subsolutions and supersolutions of (3.1).

Definition 1 *A function $\underline{Z}$ is said to be a subsolution of* (3.1) *on the whole real line if*

$$\underline{Z}_{rr} \geq \bar{H}(r, \underline{Z}, \underline{Z}_r).$$

$\bar{Z}$ is a supersolution if

$$\bar{Z}_{rr} \leq \bar{H}(r, \bar{Z}, \bar{Z}_r).$$

In addition, $(\underline{Z}, \bar{Z})$ is said to be a pair of ordered subsolution and supersolution of (3.1) *if they also satisfy*

$$\underline{Z}(r) \leq \bar{Z}(r), \quad \forall r \in \mathbf{R}.$$

We also want to define supersolutions and subsolutions of the corresponding boundary value problem on a finite interval $[r_1, r_2]$

$$\begin{cases} Z_{rr} = \bar{H}(r, Z, Z_r), \\ Z(r_1) = Z_1, \quad Z(r_2) = Z_2. \end{cases} \tag{3.2}$$

Definition 2 *A function $\underline{Z}$ is said to be a subsolution of* (3.2) *if*

$$\underline{Z}_{rr} \geq \bar{H}(r, \underline{Z}, \underline{Z}_r), \quad \underline{Z}(r_1) \leq Z_1, \quad \underline{Z}(r_2) \leq Z_2.$$

$\bar{Z}$ is a supersolution of (3.2) *if*

$$\bar{Z}_{rr} \leq \bar{H}(r, \bar{Z}, \bar{Z}_r),$$
$$\bar{Z}(r_1) \geq Z_1, \quad \bar{Z}(r_2) \geq Z_2.$$

In addition, $\underline{Z}$ and $\bar{Z}$ are said to be ordered subsolution and supersolution if they also satisfy

$$\underline{Z}(r) \leq \bar{Z}(r), \quad \forall r \in [r_1, r_2].$$

First we will show that similar existence result holds for (3.2). Then we will extend the result to the whole real line and get an existence result of (3.1). (2.22) will be a special case. The following lemma is needed.

Lemma 2 *Let $F(r, z, p)$ be continuous and bounded on $J \times \mathbf{R}^2$, where $J = [r_1, r_2]$. Then the boundary value problem*

$$\begin{cases} Z_{rr} = F(r, Z, Z_r), \\ Z(r_1) = Z_1, \quad Z(r_2) = Z_2, \end{cases}$$

has at least one solution.

Proof. This is a direct result of Walter [W] page 262 Existence Theorem XX. **Q.E.D.**

Some a priori estimates are needed to get the existence results for the boundary value problem (3.2).

Lemma 3 *Suppose $Z(r)$ is a classical $\mathbf{C}^2$ solution of (3.2) on $J = [r_1, r_2]$, and it satisfies*

$$\underline{Z}(r) \leq Z(r) \leq \bar{Z}(r) \text{ on } J,$$

where $\underline{Z}(r)$ and $\bar{Z}(r)$ are subsolution and supersolution of (3.2), respectively. Define

$$M \equiv \max \left\{ \sup_J |\bar{Z}(r)|, \sup_J |\underline{Z}(r)| \right\}. \tag{3.3}$$

Suppose that

$$|\bar{H}(r, z, p)| \leq C_1(p^2 + C_2), \tag{3.4}$$

for $r \in J$ and $|z| \leq 3M$, where M is given by (3.3) and $C_1 > 0, C_2 \geq 0$ are two constants. Then there exists a constant Λ,, which only depends on M, C_1 and C_2, such that

$$|Z_r| \leq \Lambda, \quad on \quad J.$$

Proof. Take

$$\bar{\mu} \equiv \max \left\{ 2C_1, \sqrt{C_2} \right\}.$$

Then, by the above definition, we can get that if $|p| \geq \bar{\mu}$, we have

$$|\bar{H}(r, z, p)| \leq C_1(p^2 + C_2) \leq C_1(p^2 + p^2) \leq \bar{\mu} p^2. \tag{3.5}$$

Take constants k, δ such that

$$k \geq \bar{\mu}^2 e^{2\bar{\mu}M}, \quad k\delta = e^{2\bar{\mu}M} - 1.$$

Fix an $r_0 \in [r_1, r_2]$. For $r \in [r_0, r_0 + \delta]$, define

$$w(r) \equiv \frac{1}{\bar{\mu}} \log[1 + k(r - r_0)] + Z(r_0).$$

Then we can verify that

$$\begin{aligned} w(r_0) &= Z(r_0), \\ w(r_0 + \delta) &= 2M + Z(r_0) \geq M \geq Z(r_0 + \delta), \\ |w(r)| &\leq 2M + |Z(r_0)| \leq 3M, \\ |w_r(r)| &\geq \bar{\mu}. \end{aligned}$$

Given this, noting (3.4), we can show that

$$w_{rr} - \bar{H}(r, w, w_r) \leq -\bar{\mu} w_r^2 + |\bar{H}(r, w, w_r)| \leq -\bar{\mu} w_r^2 + \bar{\mu} w_r^2 = 0.$$

Now, by virtue of Gilbarg and Trudinger [GT] Theorem 10.1 (page 263), we can get

$$w(r) \geq Z(r), \quad \forall r \in [r_0, r_0 + \delta].$$

Similarly, for $\hat{w}(r) \equiv -w(r)$, using the same method, we can get

$$-w(r) = \hat{w}(r) \leq Z(r), \quad \forall r \in [r_0, r_0 + \delta].$$

Therefore, for any $r \in (r_0, r_0 + \delta)$, we have

$$\frac{|Z(r) - Z(r_0)|}{|r - r_0|} \leq \frac{|w(r) - w(r_0)|}{|r - r_0|}.$$

Let $r \to r_0^+$, we can get

$$|Z_r(r_0)| \leq \left| \frac{k}{\bar{\mu}} \right| \equiv \Lambda.$$

Since $r_0 \in J$ is arbitrary, we are done. **Q.E.D.**

Lemma 4 *Suppose $\bar{H}(r,z,p)$ is strictly increasing with respect to z, and it satisfies* (3.4). *If $\underline{Z}$ and $\bar{Z}$ are ordered subsolution and supersolution of* (3.2) *on $J = [r_1, r_2]$, then the boundary value problem* (3.2) *has at least one solution on J such that*

$$\underline{Z}(r) \le Z(r) \le \bar{Z}(r), \ \forall r \in J.$$

Proof. Define

$$\Omega \equiv \{(r,z,p) : r \in J, \ z \in [\underline{Z}, \bar{Z}], \ |p| < \Lambda_0\}, \tag{3.6}$$

where $\Lambda_0 \equiv \max\{\Lambda, \max_J \underline{Z}_r, \max_J \bar{Z}_r\}$ and Λ is a constant as in Lemma 3.

Since $\bar{H}(r,z,p)$ is strictly increasing with respect to z, and it satisfies (3.4), it is not hard to extend $\bar{H}$ to the domain $J \times \mathbf{R}^2$, such that it is a continuous, bounded function and it is strictly increasing with respect to z. Denote the extension to be $\tilde{H}$. In addition, we can suppose that $\tilde{H}$ satisfies (3.4). For example, we can take

$$\tilde{H}_1(r,z,p) = \begin{cases} \bar{H}(r,z,p), & \text{if } r \in J, \ \underline{Z} \le z \ge \underline{Z}; \\ \bar{H}(r,\underline{Z},p) + e^z - e^{\underline{Z}}, & \text{if } r \in J, \ z < \underline{Z}; \\ \bar{H}(r,\bar{Z},p) + e^{-\bar{Z}} - e^{-z}, & \text{if } r \in J, \ z \ge \bar{Z}, \end{cases}$$

and

$$\tilde{H}(r,z,p) = \begin{cases} \tilde{H}_1(r,z,p), & \text{if } |p| \le \Lambda_0; \\ \tilde{H}_1(r,z,-\Lambda_0), & \text{if } p < -\Lambda_0; \\ \tilde{H}_1(r,z,\Lambda_0) & \text{if } p > \Lambda_0. \end{cases}$$

It is not hard to verify that $\tilde{H}(r,z,p)$ is a bounded continuous function on $J \times \mathbf{R}^2$. In addition, $\tilde{H}(r,z,p)$ is strictly increasing with respect to z and it satisfies (3.4).

Take constants Z_1, Z_2 such that

$$\underline{Z}(r_i) \le Z_i \le \bar{Z}(r_i), \quad i = 1, 2.$$

Now according to Lemma 2, we know that the boundary value problem

$$\begin{cases} Z_{rr} = \tilde{H}(r, Z, Z_r), \\ Z(r_1) = Z_1, \quad Z(r_2) = Z_2 \end{cases}$$

has a solution, say, $Z(r)$. Now we need to show that $\underline{Z} \le Z \le \bar{Z}$ and $|Z_r| \le \Lambda_0$. Assume that $Z \le \bar{Z}$ does not always hold on J. Then $\bar{Z} - Z$ is negative in an open set I_0 and is nonnegative at its endpoints. Suppose $\bar{Z} - Z$ reaches its minimum at $r_0 \in I_0$, then we have

$$\bar{Z}_r(r_0) = Z_r(r_0), \quad \bar{Z}(r_0) < Z(r_0).$$

Noting that $\tilde{H}$ is strictly increasing with respect to z, we can get

$$(\bar{Z}_{rr} - Z_{rr})(r_0) \le \tilde{H}(r_0, \bar{Z}(r_0), \bar{Z}_r(r_0)) - \tilde{H}(r_0, Z(r_0), Z_r(r_0)) < 0.$$

So $(\bar{Z} - Z)$ can not reach its minimum in I_0. This is a contradiction. Therefore, we must have $Z \le \bar{Z}$ on J. A similar argument gives $\underline{Z} \le Z$. Further, since $\tilde{H}$ satisfies (3.4), following the same procedure in the proof of Lemma 3, we can show that $|Z_r| \le \Lambda \le \Lambda_0$ on J. Therefore, we can get that $\tilde{H}(r, Z, Z_r) = \bar{H}(r, Z, Z_r)$. Therefore, Z is a solution of (3.2). **Q.E.D.**

The following uniqueness result is needed later.

Lemma 5 (Uniqueness) *Suppose $\bar{H}(r,z,p)$ is strictly increasing with respect to z, and it satisfies* (3.4). *If two* $\mathbf{C}^2$ *functions $Z(r)$ and $\tilde{Z}(r)$ are solutions of* (3.1) *on $J = [r_1, r_2]$, such that*

$$\underline{Z}(r) \le Z(r), \tilde{Z}(r) \le \bar{Z}(r), \tag{3.7}$$

then

$$Z(r) \equiv \tilde{Z}(r), \quad \text{on } J. \tag{3.8}$$

Proof. Let $\psi(r) \equiv Z(r) - \tilde{Z}(r)$. Then we have that $\psi(r_1) = \psi(r_2) = 0$. Assume that ψ reaches it minimum at $r_0 \in (r_1, r_2)$, such that $\psi(r_0) < 0$, that is, $Z(r_0) < \tilde{Z}(r_0)$, and $Z_r(r_0) = \tilde{Z}_r(r_0)$. Then, by virtue of (3.1) and the definition of ψ, noting that $\bar{H}(r,z,p)$ is strictly increasing with z, we can get

$$\psi_{rr}(r_0) < 0.$$

This contradicts the assumption that ψ reaches its minimum at $r_0 \in (r_1, r_2)$. Therefore, we must have $Z(r) \ge \tilde{Z}(r)$ on J. The same argument for $\psi = \tilde{Z} - Z$ will lead to $Z(r) \le \tilde{Z}(r)$ on J. **Q.E.D.**

Let $\big(\underline{Z}(r), \bar{Z}(r)\big)$ be a pair of ordered subsolution and supersolution of (3.1) on the whole real line, that is, $\forall r \in \mathbf{R}$,

$$\begin{aligned} \underline{Z}_{rr} &\ge \bar{H}(r, \underline{Z}, \underline{Z}_r), &(3.9)\\ \bar{Z}_{rr} &\le \bar{H}(r, \bar{Z}, \bar{Z}_r), &(3.10)\\ \underline{Z}(r) &\le \bar{Z}(r). &(3.11) \end{aligned}$$

According to the definitions, it is immediate that $\underline{Z}$ and $\bar{Z}$ are ordered subsolution and supersolution of the following problem on any $I_m \equiv [-m, m]$:

$$\begin{cases} Z_{rr} = \bar{H}(r, Z, Z_r), \\ Z(-m) = \bar{Z}(-m), \quad Z(m) = \bar{Z}(m). \end{cases} \tag{3.12}$$

Now by virtue of Lemma 4, the above problem has at least one solution $\tilde{Z}_m^0(r)$, such that

$$\underline{Z}(r) \le \tilde{Z}_m^0(r) \le \bar{Z}(r), \text{ on } I_m.$$

Define its extension on $\mathbf{R}$ by

$$\tilde{Z}_m(r) = \begin{cases} \tilde{Z}_m^0(r), & \text{if } r \in I_m, \\ \bar{Z}(r), & \text{otherwise.} \end{cases}$$

Then $\tilde{Z}_m$ is continuous. Further, we have the following lemma:

Lemma 6 *For any m, we have*

$$\underline{Z}(r) \le \tilde{Z}_{m+1}(r) \le \tilde{Z}_m(r) \le \bar{Z}(r). \tag{3.13}$$

Proof. By definition, for any m, we must have

$$\underline{Z} \le \tilde{Z}_m \le \bar{Z}.$$

So we only need to show that

$$\tilde{Z}_{m+1}(r) \le \tilde{Z}_m(r), \ \forall r. \tag{3.14}$$

By the definitions of $\{Z_m, m = 1, 2, 3, ...\}$, it is sufficient to show that the above inequality holds on I_m. Actually, it is not hard to verify that $\tilde{Z}_{m+1}$ is a subsolution of $(3.12) - (3.13)$ on I_m. Then by virtue of Lemma 4, there exists a solution $\tilde{Z}^*(r)$ of $(3.12) - (3.13)$, such that

$$\tilde{Z}_{m+1}(r) \leq \tilde{Z}^*(r) \leq \bar{Z}(r), \ \forall r \in I_m.$$

Noting the result of Lemma 5, we must have

$$\tilde{Z}^*(r) \equiv \tilde{Z}_m(r), \ \forall r \in I_m,$$

which implies that (3.14) holds on I_m. This completes our proof. **Q.E.D.**

Finally, we have the following existence result.

Theorem 1 *Suppose $\bar{H}(r, z, p)$ is strictly increasing with respect to z, and it satisfies* (3.4). *Let $\left(\underline{Z}, \bar{Z}\right)$ be a pair of ordered subsolution and supersolution of* (3.1) *on* $\mathbf{R}$. *Then* (3.1) *has a solution $Z(r)$ such that*

$$\underline{Z}(r) \leq Z(r) \leq \bar{Z}(r). \tag{3.15}$$

Proof. Consider the sequence $\{\tilde{Z}_m\}$ as in Lemma 6. It is easy to show that $\tilde{Z}_m$ converges in pointwise sense to a function Z as $m \to \infty$.

Since any bounded interval J is contained in I_m for some m, a $\mathbf{C}^2$ function Z is a solution of (3.1) if it satisfies (3.1) in I_m for any m. Let m be fixed, and let $k > m$ be arbitrary. Then for $r \in I_m$, $\tilde{Z}_k(r)$ satisfies

$$\frac{\partial^2 \tilde{Z}_k}{\partial r^2} = \bar{H}(r, \tilde{Z}_k, \frac{\partial \tilde{Z}_k}{\partial r}), \quad \tilde{Z}_k(-m) \leq \bar{Z}(-m), \quad \tilde{Z}_k(m) \leq \bar{Z}(m).$$

Since $\underline{Z} \leq \tilde{Z}_k \leq \bar{Z}$, $\forall r \in I_m$, we know that $\{\tilde{Z}_k\}$ is uniformly bounded on I_m. In addition, noting Lemma 3, we can get that $\{\frac{\partial \tilde{Z}_k}{\partial r}\}$ is uniformly bounded on I_m. Finally, by virtue of equation $(3.1), (3.4)$ and (3.15), it is not hard to show that $\{\frac{\partial^2 \tilde{Z}_k}{\partial r^2}\}$ and $\{[\frac{\partial^2 \tilde{Z}_k}{\partial r^2}]_{\alpha; I_m}\}$ are uniformly bounded on I_m.

Given the above results, using the Arzela-Ascoli theorem, we can show that $\{\tilde{Z}_m\}$ contains a subsequence which converges in $\mathbf{C}^2(I_m)$ to a function $\tilde{Z} \in \mathbf{C}^{2,\alpha}(I_m)$. Since $\{\tilde{Z}_k\}$ converges to Z in pointwise sense, $\tilde{Z}$ must coincide with Z. Moreover, the whole sequence $\{\tilde{Z}_k\}$ converges in $\mathbf{C}^2(I_m)$ to Z as $k \to \infty$. Let $k \to \infty$, and we can get that Z is a solution of (3.1) on I_m. By the arbitrariness of I_m, Z is a solution of (3.1) on $\mathbf{R}$. **Q.E.D.**

Now we only need to find a pair of ordered subsolution and supersolution to get the existence of the classical solution $\tilde{Z}(r) = Z(r)$. Then we can obtain the classical solution $\tilde{V}(x, r) = \frac{1}{\gamma} x^\gamma e^{\tilde{Z}(r)}$. This will be done for $\gamma > 0$ case in Section 4 and for $\gamma < 0$ case in Section 5. The solution will be verified to be the value function in both cases. These verification results imply that the solution $Z(r)$ to (3.1) satisfying the bounds (3.16) is unique.

It is not hard to show that the function $H(r, z, p)$ defined by (2.21) is strictly increasing with respect to z, and it satisfies (3.4). Therefore, we have the following lemma:

Lemma 7 *Let $\left(\underline{Z}, \bar{Z}\right)$ be a pair of ordered subsolution and supersolution of* (2.22) *on* $\mathbf{R}$. *Then* (2.22) *has a solution $Z(r)$ such that*

$$\underline{Z}(r) \leq Z(r) \leq \bar{Z}(r). \tag{3.16}$$

4 $\gamma > 0$ Case.

In this section, we will find a pair of ordered subsolution and supersolution when $\gamma > 0$ under some conditions, which will be specified in Lemma 8 and Lemma 9. Then we can get the existence of the solution of the reduced DPE (2.22) by using Lemma 7. Further, we need to verify that this solution is actually our value function. This result is given in Theorem 2. The admissible control space is defined by Definition 3.

Define

$$LZ = \frac{\sigma_2^2}{2} Z_{rr} + \frac{\sigma_2^2}{2}\left[1 + \frac{\gamma\rho^2}{1-\gamma}\right] Z_r^2 + \left[\frac{\gamma\rho\sigma_2(b-r)}{\sigma_1(1-\gamma)} + f(r)\right] Z_r, \tag{4.1}$$

$$h(r, Z) = [\gamma Q(r) - \beta] + (1-\gamma)e^{\frac{Z}{\gamma-1}}. \tag{4.2}$$

Then the equation (2.22) for Z can be written as

$$-LZ = h(r, Z). \tag{4.3}$$

It is easy to verify that Z is a subsolution (supersolution) of (2.22) if and only if

$$-LZ \leq (\geq) h(r, Z).$$

Lemma 8 *Supose*

$$\beta > \gamma b - \frac{\sigma_1^2}{2}\gamma(1-\gamma). \tag{4.4}$$

Define K_1 as

$$K_1 \equiv \log \tilde{K}_1, \tag{4.5}$$

where $\tilde{K}_1$ is a positive constant defined by

$$\tilde{K}_1^{\frac{1}{\gamma-1}} = \frac{1}{1-\gamma}\left[\beta - b\gamma + \frac{\sigma_1^2}{2}\gamma(1-\gamma)\right]. \tag{4.6}$$

Then, any constant $K_2 \leq K_1$ is a subsolution of (2.22).

Proof. Since K_2 is a constant, we have

$$-LK_2 = 0.$$

On the other hand, since $Q(r)$ is quadratic, by the definition of K_1, it is not hard to verify that

$$h(r, K_2) > 0,$$

for any constant $K_2 \leq K_1$. Thus, we have

$$-LK_2 < h(r, K_2).$$

Therefore, K_2 is a subsolution of (2.22). **Q.E.D.**

The constant K_1 has the following interpretation. The constant investment control $u_t = 1$ for all t (no wealth in the "riskless" asset) is suboptimal. The solution to the optimal consumption problem with this special choice for u_t has value function $\gamma^{-1}K_1 x^\gamma$. Condition (4.4) is equivalent to $K_1 > 0$.

A formal asymptotic analysis suggests (but does not prove) that $Z(r)$ in (2.20) grows quadratically as $|r| \to \infty$. With this in mind, we next seek a quadratic supersolution $\bar{Z}(r)$ of the form (4.13), where the constants a_1 and a_2 are to be suitably chosen. The bounds (4.8) on the risk sensitivity parameter γ, and the lower bound (4.14) on the discount factor β give sufficient conditions that such a supersolution $\bar{Z}(r)$ exists. Later in the section, further restrictions on a_1, a_2 and β will be imposed in order to ensure that the solution $\tilde{V}(x,r)$ to the dynamic programming equation obtained by the sub/super solution method is indeed the value function $V(x,r)$. See Theorem 3.

Lemma 9 *Define*

$$\gamma_1 \equiv \frac{\sigma_1^2 c_1^2}{\sigma_1^2 c_1^2 + \sigma_2^2 - 2c_1\rho\sigma_1\sigma_2}. \tag{4.7}$$

Assume that

$$0 < \gamma < \min\{1, \gamma_1\}. \tag{4.8}$$

In addition, define

$$\mu_1 \equiv -2\sigma_2^2\left[1 + \frac{\gamma\rho^2}{1-\gamma}\right], \tag{4.9}$$

$$\mu_2 \equiv 2c_1 + \frac{2\gamma\rho\sigma_2}{\sigma_1(1-\gamma)}, \tag{4.10}$$

$$\mu_3 \equiv -\frac{\gamma}{2\sigma_1^2(1-\gamma)}. \tag{4.11}$$

Let a^+, a^- be the real roots of $\mu_1 a^2 + \mu_2 a + \mu_3 = 0$. Then we have

$$0 < a^- < a^+. \tag{4.12}$$

Moreover, for any $a_1 \in I_1 \equiv (a^-, a^+)$, there exist constants $a_2 > K_1$ and $C_1(a_1)$, where K_1 is given by (4.5) *and $C_1(\cdot)$ are given by* (4.20)*, such that*

$$\bar{Z}(r) \equiv a_1 r^2 + a_2 \tag{4.13}$$

is a supersolution of (2.22)*, provided that*

$$\beta > -C_1(a_1). \tag{4.14}$$

Proof. Since $|\rho| \le 1$, by (4.7) we can get $\gamma_1 > 0$. Moreover, under condition (4.8), it is not hard to verify that (4.12) holds.

On the other hand, for $\bar{Z}(r)$ defined by (4.13), it is easy to verify that

$$\bar{Z}_r = 2a_1 r, \quad \bar{Z}_{rr} = 2a_1.$$

Then we have

$$-L\bar{Z} = -2a_1^2\sigma_2^2\left[1 + \frac{\gamma\rho^2}{1-\gamma}\right]r^2 - 2a_1 f(r)r - \frac{2a_1\gamma\rho\sigma_2}{\sigma_1(1-\gamma)}(b-r)r - a_1\sigma_2^2.$$

By virtue of (2.9), there exists a $\xi \in [0, r]$ such that

$$\begin{aligned} -2a_1 r f(r) &= -2a_1 r[f(0) + f_r(\xi)r] \\ &= -2a_1 f_r(\xi)r^2 - 2a_1 f(0)r \\ &\ge 2c_1 a_1 r^2 - 2a_1 f(0)r. \end{aligned}$$

Therefore, we have

$$\begin{aligned} -L\bar{Z} \ge& \left[2a_1\left(c_1 + \frac{\gamma\rho\sigma_2}{\sigma_1(1-\gamma)}\right) - 2a_1^2\sigma_2^2\left(1 + \frac{\gamma\rho^2}{1-\gamma}\right)\right]r^2 \\ &-2a_1\left[f(0) + \frac{\gamma\rho\sigma_2 b}{\sigma_1(1-\gamma)}\right]r - a_1\sigma_2^2. \end{aligned} \tag{4.15}$$

To ensure that $\bar{Z}(r)$ is a supersolution of (2.22), we only need to show that

$$-L\bar{Z} \geq h(r, \bar{Z}). \tag{4.16}$$

Define

$$\begin{aligned}
\lambda_1(a_1) &\equiv \mu_1 a_1^2 + \mu_2 a_1 + \mu_3, &(4.17)\\
\lambda_2(a_1) &\equiv -\left[2f(0) + \frac{2\gamma\rho\sigma_2}{\sigma_1(1-\gamma)}\right] a_1 + \frac{b\gamma}{\sigma_1^2(1-\gamma)} - \gamma, &(4.18)\\
\lambda_3(a_1) &\equiv -a_1\sigma_2^2 - \frac{\gamma b^2}{2\sigma_1^2(1-\gamma)}, &(4.19)\\
C_1(a_1) &\equiv \frac{4\lambda_1(a_1)\lambda_3(a_1) - \lambda_2^2(a_1)}{4\lambda_1(a_1)}, &(4.20)
\end{aligned}$$

where μ_1, μ_2, μ_3 are given by (4.9) – (4.11). Then, by virtue of (4.15), to show (4.16), it is sufficient to show that

$$\lambda_1(a_1)r^2 + \lambda_2(a_1)r + \lambda_3(a_1) \geq -\beta + (1-\gamma)e^{\frac{a_1 r^2 + a_2}{\gamma - 1}}. \tag{4.21}$$

A basic calculation implies that $\lambda_1(a_1) > 0$, provided that $a_1 \in I_1$. Then it is not hard to verify that the left hand side of (4.21) is bounded below by $C_1(a_1)$. From the definition, we know that $C_1(a_1)$ only depends on $a_1, c_1, b, \rho, \sigma_1, \sigma_2, \gamma$ and $f(0)$. Since $0 < \gamma < \min\{\gamma_1, 1\}$ and $a_1 > 0$, we have

$$e^{\frac{a_1 r^2}{\gamma-1}} \leq 1.$$

Thus, if (4.14) holds, then we can take $a_2 > K_1$ large enough such that

$$(1-\gamma)e^{\frac{a_2}{\gamma-1}} e^{\frac{a_1 r^2}{\gamma-1}} \leq (1-\gamma)e^{\frac{a_2}{\gamma-1}} \leq \beta - C_1(a_1),$$

which implies (4.21). **Q.E.D.**

Remark 3 *From* (4.7)*, we can get that* $\gamma_1 \leq 1$ *if and only if* $\sigma_2 \geq 2c_1\rho\sigma_1$.

We have the following existence results for equation (2.22):

Theorem 2 *Suppose* (4.4), (4.8) *and* (4.14) *hold. Then* (2.22) *possesses a classical solution* $\tilde{Z}(r)$ *such that*

$$K_1 \leq \tilde{Z}(r) \leq \bar{Z}(r), \tag{4.22}$$

where K_1 *and* $\bar{Z}$ *are given by* (4.5) *and* (4.13)*, respectively. Define*

$$\tilde{V}(x, r) \equiv \frac{1}{\gamma} x^\gamma e^{\tilde{Z}(r)}. \tag{4.23}$$

Then $\tilde{V}(x, r)$ *is a classical solution of* (2.14).

Proof. It is not hard to verify that $(K_1, \bar{Z}(r))$ is a pair of ordered subsolution and supersolution. Then by Lemma 7, there exists a classical solution $\tilde{Z}(r)$ of (2.22) such that (4.22) holds. By virtue of (4.23), it is not hard to verify that $\tilde{V}(x, r)$ is a classical solution of (2.14). **Q.E.D.**

Now we need to verify that $\tilde{V}(x, r)$ is equal to our value function. This will be done in Theorem 3. We will also specify the admissible control space in Definition 3. Before we go to the verification theorem, we need some lemmas. In those lemmas, we always suppose that $(r_t, t \geq 0)$ is a solution of (2.3) – (2.4).

Lemma 10 *Suppose $v(r) \in \mathbf{C}^2(\mathbf{R})$ is bounded. In addition, suppose v_r and v_{rr} are all bounded. Then $\phi(r,T) \equiv E_r e^{\int_0^T v(r_t)dt}$ is in $\mathbf{C}^{2,1}(\mathbf{R},[0,\infty))$ and it is a classical solution of*

$$\begin{cases} \phi_T = \frac{1}{2}\sigma_2^2\phi_{rr} + f(r)\phi_r + v(r)\phi, \\ \phi(r,0) = 1. \end{cases} \tag{4.24}$$

The proof is rather standard. Please refer to Pang [Pang] Lemma 1.12 for details.

Lemma 11 *Suppose $\hat{v}(r) \in \mathbf{C}^2(\mathbf{R})$. In addition, suppose $\hat{v}, \hat{v}_r, \hat{v}_{rr}$ are all bounded. Then $\eta(r,T) \equiv E_r e^{\hat{v}(r_T)}$ is in $\mathbf{C}^{2,1}(\mathbf{R},[0,\infty))$ and it is a classical solution of*

$$\begin{cases} \eta_T = \frac{1}{2}\sigma_2^2\eta_{rr} + f(r)\eta_r, \\ \eta(r,0) = e^{\hat{v}(r)}. \end{cases} \tag{4.25}$$

This is a direct corollary of Theorem 5.6.1 of A. Friedman [Fr].

Lemma 12 *Let*

$$\hat{Q}(r) \equiv \nu_1 r^2 + \nu_2 r + \nu_3, \tag{4.26}$$

where ν_1, ν_2 and ν_3 are constants and ν_1 satisfies

$$\nu_1 < \frac{c_1^2}{2\sigma_2^2}. \tag{4.27}$$

Define $\hat{a}^-, \hat{a}^+$ as the real roots of the equation $2\sigma_2^2\hat{a}^2 - 2c_1\hat{a} + \nu_1 = 0$. Then we have

$$0 < \hat{a}^- < \hat{a}^+. \tag{4.28}$$

Moreover, suppose $(r_t, t \geq 0)$ is a solution of (2.3) − (2.4) and suppose that

$$\hat{a}^- \leq \hat{a}_1 \leq \hat{a}^+. \tag{4.29}$$

Then we have

$$e^{-\beta t} E e^{\int_0^t \hat{Q}(r_s)ds} \leq \Lambda, \quad \forall t \in [0,T], \tag{4.30}$$

where Λ is a constant, provided that

$$\beta > -C_2(\hat{a}_1), \tag{4.31}$$

where $C_2(\hat{a}_1)$ is given by (4.36).

Proof. Define a sequence of functions $\{\hat{Q}_M(r), M = 1,2,3,...\}$ such that

$$\hat{Q}_M \in \mathbf{C}^\infty; \quad 0 \leq \hat{Q}_M \leq M; \quad \left|\frac{\partial \hat{Q}_M(r)}{\partial r}\right| \leq \hat{M}; \quad \left|\frac{\partial^2 \hat{Q}_M(r)}{\partial r^2}\right| \leq \hat{M};$$
$$\hat{Q}_{M_1}(r) \leq \hat{Q}_{M_2}(r) \leq \hat{Q}(r),\ M_1 < M_2; \quad \lim_{M\to\infty} \hat{Q}_M(r) = \hat{Q}(r),$$

where $\hat{M}, M_1, M_2$ are constants. Define

$$\psi(r,t) \equiv e^{-\beta t} E e^{\int_0^t \hat{Q}_M(r_s)ds}.$$

Then according to Lemma 10, $\psi \in \mathbf{C}^{2,1}(\mathbf{R},[0,\infty))$ and it is a solution of the problem

$$\begin{cases} \frac{\partial\psi}{\partial t} = \frac{\sigma_2^2}{2}\frac{\partial^2\psi}{\partial r^2} + f(r)\frac{\partial\psi}{\partial r} + [\hat{Q}_M(r) - \beta]\psi, \\ \psi(r,0) = 1. \end{cases} \tag{4.32}$$

It is easy to verify that, under condition (4.27), $\hat{a}^-$ and $\hat{a}^+$ are real, positive numbers. So we can take an $\hat{a}_1$ such that (4.29) holds.

Define

$$\begin{aligned}
\hat{\lambda}_1(\hat{a}_1) &= -2\sigma_2^2\hat{a}_1^2 + 2c_1\hat{a}_1 - \nu_1, && (4.33)\\
\hat{\lambda}_2(\hat{a}_1) &= -2\hat{a}_1 - \nu_2, && (4.34)\\
\hat{\lambda}_3(\hat{a}_1) &= -\sigma_2^2\hat{a}_1 - \nu_3, && (4.35)\\
C_2(\hat{a}_1) &= \frac{4\hat{\lambda}_1(\hat{a}_1)\hat{\lambda}_3(\hat{a}_1) - \hat{\lambda}_2(\hat{a}_1)}{4\hat{\lambda}_1(\hat{a}_1)}. && (4.36)
\end{aligned}$$

Following the same procedure in the proof of Lemma 9, it is not hard to verify that, under conditions(4.29) and (4.31), $\bar{\psi}(r) \equiv e^{\hat{a}_1 r^2}$ satisfies

$$\begin{cases} \frac{\partial\bar{\psi}}{\partial T} \geq \frac{\sigma_2^2}{2}\frac{\partial^2\bar{\psi}}{\partial r^2} + f(r)\frac{\partial\bar{\psi}}{\partial r} + [\hat{Q}_M(r) - \beta]\bar{\psi}, \\ \bar{\psi}(r,0) \geq 1. \end{cases}$$

Define $\xi(r,T) \equiv \psi(r,T) - \bar{\psi}(r)$. Then it satisfies

$$\begin{cases} \frac{\partial\xi}{\partial T} \leq \frac{\sigma_2^2}{2}\frac{\partial^2\xi}{\partial r^2} + f(r)\frac{\partial\xi}{\partial r} + [\hat{Q}_M(r) - \beta]\xi, \\ \xi(r,0) \leq 0. \end{cases}$$

Since $\hat{Q}_M$ is bounded, there exists a constant $B > 0$ such that $\hat{Q}_M(r) - \beta < B$. Define

$$\tilde{\xi}(r,T) \equiv e^{-BT}\xi(r,T), \quad \tilde{Q}_M(r) \equiv \hat{Q}_M(r) - \beta - B.$$

Then $\tilde{Q}_M(r) < 0$ and $\tilde{\xi}$ satisfies

$$\begin{cases} \frac{\partial\tilde{\xi}}{\partial T} \leq \frac{\sigma_2^2}{2}\frac{\partial^2\tilde{\xi}}{\partial r^2} + f(r)\frac{\partial\tilde{\xi}}{\partial r} + \tilde{Q}_M(r)\tilde{\xi}, \\ \tilde{\xi}(r,0) \leq 0. \end{cases}$$

Since $\hat{Q}_M(r)$ is bounded, by definitions of $\psi, \bar{\psi}, \xi$ and $\tilde{\xi}$, we can get

$$\lim_{|r|\to\infty} \tilde{\xi}(r,T) = -\infty.$$

If $\tilde{\xi}(r,T)$ reaches its maximum on $\mathbf{R}\times[0,T_1]$ at a point (r_0,T_0), such that $\tilde{\xi}(r_0,T_0) > 0$, then we must have $T_0 > 0$ and

$$\tilde{\xi}_r(r_0,T_0) = 0, \quad \tilde{\xi}_{rr}(r_0,T_0) \leq 0, \quad \tilde{\xi}_T(r_0,T_0) \geq 0.$$

This contradicts

$$\frac{\partial\tilde{\xi}}{\partial T} \leq \frac{\sigma_2^2}{2}\frac{\partial^2\tilde{\xi}}{\partial r^2} + f(r)\frac{\partial\tilde{\xi}}{\partial r} + \tilde{Q}_M(r)\tilde{\xi}.$$

Therefore, we must have

$$\tilde{\xi}(r,T) \leq 0, \quad \forall r, T.$$

By definitions of $\tilde{\xi}$ and ξ, we can get

$$\psi(r,T) \leq \bar{\psi}(r), \quad \forall r, T.$$

Define $\Lambda \equiv \bar{\psi}(r)$. Then Λ is a constant which does not depend on $M, \hat{M}$ or T. Thus, by the Monotone Convergence Theorem, we can get (4.30). **Q.E.D.**

Lemma 13 *Define*

$$\gamma_2 \equiv \frac{\sigma_1^2 c_1^2}{2\sigma_1^2 c_1^2 + 4\sigma_2^2}, \tag{4.37}$$

and suppose that

$$0 < \gamma < \gamma_2. \tag{4.38}$$

Define

$$I_2 \equiv \left(\frac{c_1}{2\sigma_2^2} - \frac{1}{2\sigma_2^2}\sqrt{c_1^2 - \frac{4\sigma_2^2\gamma}{\sigma_1^2(1-2\gamma)}},\ \frac{c_1}{2\sigma_2^2} + \frac{1}{2\sigma_2^2}\sqrt{c_1^2 - \frac{4\sigma_2^2\gamma}{\sigma_1^2(1-2\gamma)}} \right), \tag{4.39}$$

and assume that

$$\hat{a}_2 \in I_2. \tag{4.40}$$

Define

$$\nu_1 = \frac{2\gamma}{\sigma_1^2(1-2\gamma)}, \quad \nu_2 = -\frac{4b\gamma}{\sigma_1^2(1-2\gamma)} + 4\gamma, \quad \nu_3 = \frac{2b^2\gamma}{\sigma_1^2(1-2\gamma)}. \tag{4.41}$$

Then for any

$$\beta > -\frac{1}{2}C_2(\hat{a}_2), \tag{4.42}$$

where $C_2(\hat{a}_2)$ is given by (4.36) with ν_1, ν_2, ν_3 defined above, there is a constant Λ, which is independent of T, such that

$$e^{-2\beta T} E_r e^{\int_0^T 4\gamma Q_1(r_t)dt} \le \Lambda, \tag{4.43}$$

where

$$Q_1(r) \equiv \frac{(b-r)^2}{2\sigma_1^2(1-2\gamma)} + r. \tag{4.44}$$

Proof. This is a direct corollary of Lemma 12. **Q.E.D.**

Lemma 14 *Define*

$$I_3 \equiv \left(0, \frac{c_1}{K\sigma_2^2}\right), \tag{4.45}$$

where $K > 8$ is a constant. Assume that

$$a_3 \in I_3. \tag{4.46}$$

Define

$$C_3(a_3) \equiv \frac{Kf(0)^2 a_3}{2K\sigma_2^2 a_3 - 2c_1} - K\sigma_2^2 a_3. \tag{4.47}$$

Then for any

$$\beta > -\frac{1}{2}C_3(a_3), \tag{4.48}$$

there is a constant Λ, which is independent of T, such that

$$e^{-2\beta T} E_r e^{K a_3 r_T^2} \le \Lambda.$$

The proof is almost the same as the proof of Lemma 12. So we omit it here. Refer to Pang [Pang] Lemma 1.15 for details.

Lemma 15 *Suppose* (4.8) *holds. Define*

$$k_1 \equiv \tfrac{K\rho\sigma_2}{\sigma_1} - 2c_1\rho^2,\ k_2 \equiv \tfrac{K^2(2c_1\rho\sigma_1\sigma_2-\sigma_2^2)}{\sigma_1^2},\ \bar{\nu}_1 \equiv \tfrac{(2-K)c_1}{k_1},\ \gamma_3 \equiv \tfrac{\bar{\nu}_1}{1+\bar{\nu}_1}, \qquad (4.49)$$

$$\bar{\nu}_2 \equiv \tfrac{-2(K-2)c_1k_1+k_2}{2k_1} + \tfrac{\sqrt{(2(K-2)c_1k_1-k_2)^2+4(4K-4)c_1^2k_1^2}}{2k_1^2}, \quad \gamma_4 \equiv \tfrac{\bar{\nu}_2}{1+\bar{\nu}_2}, \qquad (4.50)$$

where K *is the constant in Lemma 14. Then if*

$$k_1 < 0, \quad 0 < \gamma < \gamma_3, \qquad (4.51)$$

or

$$k_1 > 0, \quad 0 < \gamma < \gamma_4, \qquad (4.52)$$

we have

$$I_1 \cap I_3 \neq \emptyset. \qquad (4.53)$$

Proof. It can be verified by virtue of some basic calculations. For details, please see the proof of Lemma 1.16 in Pang [Pang]. **Q.E.D.**

Definition 3 (Admissible Control Space) *The admissible control space* Π *is*

$$\Pi \equiv \left\{(u_t, c_t) : P\left(\int_0^T u_t^2 dt < \infty\right) = 1,\ \forall T > 0,\ c_t \geq 0\right\}. \qquad (4.54)$$

We have the following lemma:

Lemma 16 *For* $(u_t, c_t) \in \Pi$, *define*

$$Y_t \equiv e^{2\gamma\sigma_1 \int_0^t u_s dw_{1,s} - 2\gamma^2\sigma_1^2 \int_0^t u_s^2 ds}, \qquad (4.55)$$

and define τ_R *to be the exit time of* (x_t, r_t) *from the ball* $\{x^2 + r^2 \leq R^2\}$. *Then we have*

$$EY_{T\wedge\tau_R} \leq 1, \quad \forall T > 0. \qquad (4.56)$$

Proof. Since $(u_t, c_t) \in \Pi$, we can get that $P(Y_t < \infty) = 1$. Then, by virtue of Ito's rule, we can get

$$Y_{T\wedge\tau_R} = 1 + 2\gamma\sigma_1 \int_0^{T\wedge\tau_R} u_s Y_s dw_{1,s}.$$

Denote $\tau_n \equiv \inf\left\{t \leq T : \int_0^t u_s^2 Y_s^2 ds \geq n^2\right\}$. Then it is easy to verify that $Y_{T\wedge\tau_R\wedge\tau_n}$ is a martingale and for any $n > 0$, and it satisfies $EY_{T\wedge\tau_R\wedge\tau_n} = 1$. Since Y_t is non-negative, by virtue of Fatou's lemma, we can get (4.56). **Q.E.D.**

Theorem 3 *Suppose that* (4.4), (4.8), (4.38) *hold and either* (4.51) *or* (4.52) *holds. In addition, assume*

$$a_1 \in I_1 \cap I_3, \quad a_2 \in I_2 \qquad (4.57)$$

and

$$\beta > \max\{-C_1(a_1), -\frac{1}{2}C_2(a_2), -\frac{1}{2}C_3(a_1)\}, \qquad (4.58)$$

where $C_1(\cdot), C_2(\cdot)$ *and* $C_3(\cdot)$ *are given by* (4.20), (4.36) *and* (4.47), *respectively.*

Define $V(x,r)$ *as in* (2.12) *and define* $\tilde{V}(x,r), \tilde{Z}(r)$ *as in Theorem 2. Then we have*

$$\tilde{V}(x,r) \equiv V(x,r). \qquad (4.59)$$

In addition, $J(x, r, u_\cdot, c_\cdot)$ reaches its maximum at

$$u^*(r) = \frac{(b-r)}{\sigma_1^2(1-\gamma)} + \frac{\rho\sigma_2 \tilde{Z}_r(r)}{\sigma_1(1-\gamma)}, \quad c^*(r) = e^{\frac{\tilde{Z}(r)}{\gamma-1}}. \tag{4.60}$$

Proof. For any admissible control $(u_t, c_t) \in \Pi$, denote $\mathcal{G}^{u_t,c_t}$ as the generator of the process (x_t, r_t) under control (u_t, c_t). Then, by virtue of the Ito's rule, we can get

$$\begin{aligned} d\left[e^{-\beta t}\tilde{V}(x_t, r_t)\right] &= e^{-\beta t}\left[d\tilde{V}(x_t, r_t) - \beta\tilde{V}(x_t, r_t)dt\right] \\ &= e^{-\beta t}\left[\mathcal{G}^{u_t,c_t}\tilde{V}(x_t, r_t) - \beta\tilde{V}(x_t, r_t)\right]dt + dm_{1,t} + dm_{2,t}, \end{aligned}$$

where $m_{1,t}$ and $m_{2,t}$ are local martingales under P.

Integrate it on $[0, T]$. Since $\tilde{V}$ is a classical solution of (2.14), we have

$$e^{-\beta T}\tilde{V}(x_T, r_T) - \tilde{V}(x, r) \leq -\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt + m_{1,T} + m_{2,T}.$$

Let τ_R define the exit time of (x_t, r_t) from the ball $\{x^2 + r^2 < R^2\}$. Then, for every finite T, we have

$$\tilde{V}(x, r) \geq E\int_0^{T\wedge\tau_R} e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt + E\left[e^{-\beta T\wedge\tau_R}\tilde{V}(x_{T\wedge\tau_R}, r_{T\wedge\tau_R})\right]. \tag{4.61}$$

Noting $\tilde{V} > 0$, by virtue of Fatou's lemma as $R \to \infty$, we can take lim inf to get

$$\tilde{V}(x, r) \geq E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt.$$

Now, let $T \to \infty$, then we have

$$\tilde{V}(x, r) \geq E\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt,$$

which holds for any admissible control $(u_t, c_t) \in \Pi$. By the definition of $V(x, r)$, we must have, for any r,

$$\tilde{V}(x, r) \geq V(x, r). \tag{4.62}$$

On the other hand, for control (u^*, c^*) defined by (4.60), it is not hard to verify that $(u_t^*, c_t^*) \in \Pi$. Then, instead of (4.61), we can get

$$\tilde{V}(x, r) = E\int_0^{T\wedge\tau_R} e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt + E\left[e^{-\beta T\wedge\tau_R}\tilde{V}(x_{T\wedge\tau_R}, r_{T\wedge\tau_R})\right]. \tag{4.63}$$

Using the Monotone Convergence Theorem, for any fixed $T > 0$, we can show that

$$\lim_{R\to\infty} E\int_0^{T\wedge\tau_R} e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt = E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt. \tag{4.64}$$

Define $\tilde{W}(r) \equiv e^{\tilde{Z}(r)}$. Then by virtue of (4.23), we can rewrite $\tilde{V}(x, r)$ as

$$\tilde{V}(x, r) \equiv \frac{1}{\gamma}x^\gamma\tilde{W}(r), \tag{4.65}$$

Define

$$\tilde{l}(r,u) = r + (b-r)u - \frac{1}{2}\sigma_1^2 u^2.$$

Then for any admissible control $(u_t, c_t) \in \Pi$, using Ito's formula, we can get

$$x_t = x \exp\left\{ \int_0^t [\tilde{l}(r_s, u_s) - c_s]ds + \int_0^t \sigma_1 u_s dw_{1,s} \right\}.$$

Given the above equality, by virtue of the Cauchy-Schwarz Inequality, we can get

$$\begin{aligned} & E\left[e^{-\beta T\wedge\tau_R}\tilde{V}(x_{T\wedge\tau_R}, r_{T\wedge\tau_R})\right] \\ = & \frac{1}{\gamma}E\left[e^{-\beta T\wedge\tau_R}x^{\gamma}_{T\wedge\tau_R}\tilde{W}(r_{T\wedge\tau_R})\right] \\ \leq & \frac{1}{\gamma}x^{\gamma}\left(Ee^{\int_0^{T\wedge\tau_R}2[\gamma l(r_s,u_s)-\gamma c_s-\beta]ds}\tilde{W}^2(r_{T\wedge\tau_R})\right)^{\frac{1}{2}} \\ & \cdot\left(Ee^{\int_0^{T\wedge\tau_R}[2\gamma\sigma_1 u_t dw_{1,t} - 2\gamma^2\sigma_1^2 u_t^2 dt]}\right)^{\frac{1}{2}}, \end{aligned}$$

where

$$l(r,u) \equiv r + (b-r)u + \left(\gamma - \frac{1}{2}\right)\sigma_1^2 u^2. \tag{4.66}$$

Using the result of Lemma 16, we have

$$Ee^{\int_0^{T\wedge\tau_R}[2\gamma\sigma_1 u_t dw_{1,t} - 2\gamma^2\sigma_1^2 u_t^2 dt]} \leq 1. \tag{4.67}$$

Thus, we can get

$$\begin{aligned} \tilde{V}(x,r) = & E\int_0^{T\wedge\tau_R} e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^{\gamma}dt \\ & + \frac{1}{\gamma}x^{\gamma}\left(Ee^{\int_0^{T\wedge\tau_R}2[\gamma l(r_s,u_s)-\gamma c_s-\beta]ds}\tilde{W}^2(r_{T\wedge\tau_R})\right)^{\frac{1}{2}}. \end{aligned} \tag{4.68}$$

From (4.37), we can get that $\gamma_2 < \frac{1}{2}$. Since $0 < \gamma < \gamma_2 < \frac{1}{2}$, by virtue of (4.44) and (4.66), we can get

$$l(r,u) \leq Q_1(r).$$

Therefore, for $0 < \gamma < \gamma_2$, $\gamma Q_1(r) - \beta$ is lower bounded. Choose B such that $\gamma Q_1(r) - \beta - B \geq 0$. Noting that $c^* > 0, \tilde{W}(r) \leq e^{a_1 r^2 + a_2}$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left[e^{\int_0^{T\wedge\tau_R}2[\gamma l(r_s,u_s^*)-\gamma c_s^*-\beta]ds}\tilde{W}^2(r_{T\wedge\tau_R})\right] \\ \leq & \left[e^{\int_0^{T\wedge\tau_R}2[\gamma Q_1(r_s)-\beta]ds}\tilde{W}^2(r_{T\wedge\tau_R})\right] \\ = & e^{2B(T\wedge\tau_R)}\left[e^{\int_0^{T\wedge\tau_R}2[\gamma Q_1(r_s)-\beta-B]ds}\sup_{0\leq t\leq T}\tilde{W}^2(r_t)\right] \\ \leq & e^{2B(T\wedge\tau_R)}\left[e^{\int_0^{T}2[\gamma Q_1(r_s)-\beta-B]ds}\sup_{0\leq t\leq T}\tilde{W}^2(r_t)\right] \\ = & e^{2B(T\wedge\tau_R-T)}\left[e^{\int_0^{T}2[\gamma Q_1(r_s)-\beta]ds}\sup_{0\leq t\leq T}\tilde{W}^2(r_t)\right] \end{aligned}$$

$$\begin{aligned}
&\le e^{2|B|T}\left[e^{\int_0^T 2[\gamma Q_1(r_s)-\beta]ds}\sup_{0\le t\le T}\tilde{W}^2(r_t)\right]\\
&\le \frac{1}{2}e^{2(|B|-\beta)T}\left[e^{\int_0^T 4\gamma Q_1(r_s)ds}+\sup_{0\le t\le T}\tilde{W}^4(r_t)\right]\\
&\le \frac{1}{2}e^{2(|B|-\beta)T}\left[e^{\int_0^T 4\gamma Q_1(r_s)ds}+A^2\sup_{0\le t\le T}e^{4a_1r_t^2}\right]\\
&\equiv \eta_T,
\end{aligned}$$

where

$$A\equiv e^{2a_2}.$$

Next, we are going to show that

$$E_r\eta_T<\infty, \tag{4.69}$$

which ensures that we can use Fatou's lemma in (4.68) to get rid of the stopping time.

First, by virtue of Lemma 13, we have

$$E_re^{\int_0^T 4\gamma Q_1(r_s)ds}<\infty \tag{4.70}$$

provided that $0<\gamma<\gamma_2$. Define $\psi(r_t)\equiv e^{4a_1r_t^2}$. Then we also need to show that

$$E_r\left[\sup_{0\le t\le T}\psi(r_t)\right]<\infty. \tag{4.71}$$

By virtue of the Ito's lemma, we have

$$\begin{aligned}
d\psi(r_t) &= 8a_1r_t\psi(r_t)[f(r_t)dt+\sigma_2dw_{2,t}]+\frac{\sigma_2^2}{2}[8a_1\psi(r_t)+64a_1^2r_t^2\psi(r_t)]dt\\
&\le [8a_1(4a_1\sigma_2^2-c_1)r_t^2+8a_1f(0)r_t+4a_1\sigma_2^2]\psi(r_t)dt+dm_t,
\end{aligned}$$

where

$$m_t\equiv 8a_1\sigma_2\int_0^t r_s\psi(r_s)dw_{2,s}. \tag{4.72}$$

By the definition of a_1, we know that $4a_1\sigma_2^2-c_1<0$. Therefore, $8a_1(4a_1\sigma_2^2-c_1)r_t^2+8a_1f(0)r_t+4a_1\sigma_2^2$ is upper-bounded. Suppose $N>0$ is an upper bound, then we have

$$\psi(r_t)\le\psi(r)+\int_0^t N\psi(r_s)ds+m_t. \tag{4.73}$$

By the definition of m_t, we have

$$E_rm_t^2=64a_1^2\sigma_2^2E\int_0^t r_s^2\psi^2(r_s)ds.$$

For any $s\in[0,t]$, by virtue of Lemma 14, we have

$$E_r\left[r_s^2\psi^2(r_s)\right]=E_r\left[r_s^2e^{8a_1r_s^2}\right]\le\Lambda_1[E_re^{(8+\epsilon)a_1r_s^2}]<\infty, \tag{4.74}$$

where $\Lambda_1>0$ is a constant and ϵ can be any positive number. Therefore, we must have $E_rm_t^2<\infty$. So m_t is a martingale. Using fundamental martingale inequalities, we can show that

$$E_r\left[\sup_{0\le t\le T}m_t^2\right]\le 4E_rm_T^2\le\Lambda_2<\infty, \tag{4.75}$$

where Λ_1 is a positive constant. Given this and (4.73), using the Chebyshev Inequality, we can show that

$$E_r\left[\sup_{0\le t\le T}\psi(r_t)\right]\le \psi(r)+\Lambda_2+\int_0^T NE_r\left[\sup_{0\le s\le t}\psi(r_s)\right]dt. \tag{4.76}$$

Then, by virtue of the Gronwall's Inequality, it is easy to get (4.71). Combined with (4.70), this implies (4.69). Now, when we let $R\to\infty$ and take $\limsup$ in (4.68), we can use the Monotone Convergence Theorem and Fatou's lemma to get

$$\tilde V(x,r)\le E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt+\frac{1}{\gamma}x^\gamma\left(E\left[e^{\int_0^T 2[\gamma Q_1(r_s)-\gamma c_s^*-\beta]ds}\tilde W^2(r_T)\right]\right)^{\frac12}. \tag{4.77}$$

Denote $\delta=\beta-\max\{-C_1(a_1),C_2(\tilde a),C_3(a_1)\}$. Then, from (4.58), we can get that $\delta>0$. Take $\hat\beta=\beta-\frac{\delta}{2}$. Then, by virtue of $c^*\ge 0$, using the Cauchy-Schwarz Inequality, we can get

$$\begin{aligned}
& E_r\left[e^{\int_0^T 2[\gamma Q_1(r_s)-\gamma c_s^*-\beta]ds}\tilde W^2(r_T)\right]\\
< \;& e^{-2\beta T}\left[E_r e^{\int_0^T 4\gamma Q_1(r_s)ds}\right]^{\frac12}\cdot\left[E_r\tilde W^4(r_T)\right]^{\frac12}\\
\le \;& e^{-\delta T}e^{4a_2}\left[e^{-2\hat\beta T}E_r e^{\int_0^T 4\gamma Q_1(r_s)ds}\right]^{\frac12}\left[e^{-2\hat\beta T}E_r e^{4a_1 r_T^2}\right]^{\frac12}
\end{aligned}$$

Given above inequality, by virtue of Lemma 13 and Lemma 14, we can show that, for any fixed $T>0$,

$$\lim_{T\to\infty}E_r\left[e^{\int_0^T 2[\gamma Q(r_s)-\gamma c_s^*-\beta]ds}\tilde W^2(r_T)\right]=0.$$

Now in (4.77), let $T\to\infty$, then we have

$$\tilde V(x,r)\le E\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt\le V(x,r). \tag{4.78}$$

Combined with (4.62), this implies

$$\tilde V(x,r)=E\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt=V(x,r).$$

Thus, (u^*,c^*) is optimal and $\tilde V(x,r)\equiv V(x,r)$. **Q.E.D.**

5 $\gamma < 0$ Case

In this section, we will investigate $\gamma < 0$ case. The existence results will be given in Theorem 4 and the verification results will be given in Theorem 5. The admissible control space will be specified in Definition 4.

Using the same notations as in last section, we can write the equation of $Z(r)$ as

$$-LZ = h(r, Z). \tag{5.1}$$

For $\gamma < 0$ case, we have the following results. A formal asymptotic analysis suggests that, for $\gamma < 0$, $Z(r)$ in (2.20) behaves like $2(\gamma-1)\log r$ as $|r| \to \infty$. This leads to the choice of $\underline{Z}(r)$ in (5.3) and the choice of $\bar{Z}(r)$ in (5.7). Condition (5.6) is sufficient for the existence of supersolution of the form (5.7).

Lemma 17 *Suppose $\gamma < 0$. Define*

$$a_1 \equiv \frac{-2\gamma}{3\sigma_1^2(1-\gamma)^2}, \qquad a_2 \equiv b - \sigma_1^2(1-\gamma). \tag{5.2}$$

Then there exists a constant $\bar{a}_3 > 0$ such that for any $a_3 \geq \bar{a}_3$

$$\underline{Z}(r) \equiv \log\left[(a_1(r-a_2)^2 + a_3)^{\gamma-1}\right] \tag{5.3}$$

is a subsolution of (5.1)*.*

It can be verified by virtue of direct calculations. See Pang [Pang] Lemma 1.18 for details.

Lemma 18 *Suppose $\gamma < 0$. Define*

$$\begin{aligned} b_1 &\equiv \frac{-\gamma}{2\sigma_1^2(1-\gamma)^2}, \qquad b_2 \equiv b - \sigma_1^2(1-\gamma), &(5.4)\\ b_3 &\equiv b_1 \frac{2\sigma_2^2[\frac{3}{2} - \gamma + \gamma\rho^2] - 2\rho\sigma_1^3\sigma_2\gamma(1-\gamma) + |f(b_2)|}{2c_2 + |f(b_2)|}. &(5.5)\end{aligned}$$

If

$$\begin{aligned}\beta \;\geq\; & b\gamma + (1-\gamma)\left[2c_2|f(b_2)| - \frac{\sigma_1^2\gamma}{2}\right] \\ & - \frac{2\gamma\sigma_2^2[\frac{3}{2} - \gamma + \gamma\rho^2] - 2\rho\sigma_1^3\sigma_2\gamma^2(1-\gamma) + \gamma|f(b_2)|}{2\sigma_1^2(1-\gamma)[2c_2 + |f(b_2)|]},\end{aligned} \tag{5.6}$$

Then

$$\bar{Z}(r) \equiv \log\left[(b_1(r-b_2)^2 + b_2)^{\gamma-1}\right] \tag{5.7}$$

is a supersolution of (5.1)*.*

The proof involves a lot of calculations. The techniques used in the proof are very similar to the proof in Lemma 9. Please see Pang [Pang] Lemma 1.19 for details.

Remark 4 *From* (5.2), (5.3), (5.6) *and* (5.7) *we can see that*

$$a_1 > b_1 \quad a_2 = b_2, \tag{5.8}$$

In addition, we can take a_3 large enough such that

$$a_3 > b_3. \tag{5.9}$$

Then

$$\bar{Z}(r) > \underline{Z}(r), \quad \forall r. \tag{5.10}$$

Given the above results, we can get

Theorem 4 *Suppose $\gamma < 0$ and (5.6) holds. Then (5.1) possesses a classical solution $\tilde{Z}(r)$ such that*

$$\underline{Z}(r) \le \tilde{Z}(r) \le \bar{Z}(r), \tag{5.11}$$

where $\underline{Z}(r)$ and $\bar{Z}(r)$ are given by (5.3) and (5.7), respectively. Define

$$\tilde{V}(x,r) \equiv \frac{1}{\gamma} x^{\gamma} e^{\tilde{Z}(r)}. \tag{5.12}$$

Then $\tilde{V}(x,r)$ is a classical solution of (2.14), and it satisfies

$$\frac{1}{\gamma} x^{\gamma} e^{\underline{Z}(r)} \le \tilde{V}(x,r) \le \frac{1}{\gamma} x^{\gamma} e^{\bar{Z}(r)}. \tag{5.13}$$

The proof is almost the same as the proof of Theorem 2. So we omit it here.

Lemma 19 *Suppose $\gamma < 0$. Let $\tilde{Z}(r)$ is a solution of (5.1) which satisfies (5.11). Then we have*

$$\lim_{|r|\to\infty} \tilde{Z}_r(r) = 0. \tag{5.14}$$

Proof. By the definitions of $\bar{Z}$ and $\underline{Z}$, we can get

$$(\gamma - 1)\log(a_1(r-a_2)^2 + a_3) \le \tilde{Z}(r) \le (\gamma - 1)\log(b_1(r-b_2)^2 + b_3). \tag{5.15}$$

The above inequality implies

$$\liminf_{|r|\to\infty} |\tilde{Z}_r(r)| = 0. \tag{5.16}$$

Otherwise, $\tilde{Z}(r)$ will have at least a linear growth as $|r| \to \infty$, which contradicts (5.15).

If (5.14) does not hold, $\tilde{Z}_r(r)$ must have a sequence of either positive local maxima or negative local minima at points $\{r_m, m = 1,2,3,...\}$, which tend to either $+\infty$ or $-\infty$ with the following property: there exists $\delta > 0$, such that

$$|\tilde{Z}_r(r_m)| \ge \delta. \tag{5.17}$$

Suppose that $\tilde{Z}_r(r)$ has a positive local maximum at r_m. Since $\tilde{Z}(r) \in \mathbf{C}^2(\mathbf{R})$, by virtue of (5.1), we have that $\tilde{Z}(r) \in \mathbf{C}^3(\mathbf{R})$. Therefore, $\tilde{Z}_{rr}(r_m) = 0$, $\tilde{Z}_{rrr}(r_m) \le 0$. Define

$$\hat{\sigma}_2^2 \equiv \sigma_2^2\left(1 + \frac{\gamma\rho}{1-\gamma}\right), \quad \hat{f}(r) \equiv f(r) + \frac{\gamma\rho\sigma_2(b-r)}{\sigma_1(1-\gamma)}, \quad \hat{c}_1 \equiv c_1 + \frac{\gamma\rho\sigma_2}{\sigma_1(1-\gamma)}.$$

Noting (2.9), we can get that

$$\hat{f}_r(r) \le -\hat{c}_1. \tag{5.18}$$

By virtue of (5.1), we can get

$$0 = \frac{\sigma_2^2}{2}\tilde{Z}_{rrr} + \hat{\sigma}_2^2 \tilde{Z}_r \tilde{Z}_{rr} + \hat{f}\tilde{Z}_{rr} + \hat{f}_r \tilde{Z}_r + \gamma Q_r - e^{-\frac{\tilde{Z}}{1-\gamma}}\tilde{Z}_r,$$

where $Q(r)$ is defined by (2.19), and Q_r stands for its derivative. Since $\tilde{Z}_{rr}(r_m) = 0$, $\tilde{Z}_{rrr}(r_m) \le 0$, noting that $\tilde{Z} \le \bar{Z}$, we have

$$0 \le (\hat{f}_r(r_m) - b_1(r_m - b_2)^2 - b_3)\tilde{Z}_r(r_m) + \gamma\left(1 + \frac{r_m - b}{\sigma_2^2(1-\gamma)}\right). \tag{5.19}$$

If $|r_m|$ is big enough, by virtue of (5.18), we will have

$$-\hat{f}_r(r_m) + b_1(r_m - b_2)^2 + b_3 \geq \hat{c}_1 + b_1(r_m - b_2)^2 + b_3 > 0.$$

Therefore, by virtue of (5.19), we can get

$$\tilde{Z}_r(r_m) \leq \frac{\left|\gamma\left(1 + \frac{r_m - b}{\sigma_2^2(1-\gamma)}\right)\right|}{\hat{c}_1 + b_1(r_m - b_2)^2 + b_3}.$$

But the right hand side of the above inequality goes to 0 as $|r_m|$ goes to $+\infty$, so we must have

$$\lim_{|r_m|\to\infty} \tilde{Z}_r(r_m) = 0. \tag{5.20}$$

This contradicts our assumption (5.17). Similarly, if $\tilde{Z}_r$ has a negative local minimum at $\{r_m, m = 1, 2, ...\}$, and we can also get (5.20). Therefore, (5.17) holds. **Q.E.D.**

Define the admissible control space Π as follows:

Definition 4 (Admissible Control Space) *A control $(u_t, c_t) \in \mathbf{R}^2$ is in the admissible control space Π, if the following hold:*

$$0 \leq c_t \leq A_1(r_t - A_2)^2 + A_3, \quad \forall t \geq 0, \tag{5.21}$$

$$E\int_0^T u_t^2 dt < \infty, \quad \forall T \geq 0, \tag{5.22}$$

$$E\int_0^T e^{-2\beta t} u_t^2 x_t^{2\gamma} dt < \infty, \quad \forall T \geq 0, \tag{5.23}$$

where $A_1 > 0, A_3 > 0$ and A_2 are some constants.

We have the following verification theorem:

Theorem 5 *Suppose $\gamma < 0$ and (5.6) holds. In addition, assume that*

$$\frac{4\gamma(1+3\gamma)}{\sigma_1^2(1-\gamma)^2} \leq \frac{c_1^2}{2\sigma_2^2}. \tag{5.24}$$

Define $V(x,r)$ as in (2.12) and define $\tilde{V}(x,r)$ and $\tilde{Z}(r)$ as in Theorem 4. Then we have

$$\tilde{V}(x,r) \equiv V(x,r). \tag{5.25}$$

In addition, $J(x, r, u_\cdot, c_\cdot)$ reaches its maximum at

$$u^*(r) = \frac{(b-r)}{\sigma_1^2(1-\gamma)} + \frac{\rho\sigma_2\tilde{Z}_r(r)}{\sigma_1(1-\gamma)}, \quad c^*(r) = e^{\frac{\tilde{Z}(r)}{\gamma-1}}. \tag{5.26}$$

Proof. For any admissible control $(u_t, c_t) \in \Pi$, denote $\mathcal{G}^{u_t,c_t}$ as the generator of the process (x_t, r_t) under control (u_t, c_t). Then, by Ito's rule, we can get

$$d\left[e^{-\beta t}\tilde{V}(x_t, r_t)\right] = e^{-\beta t}\left[\mathcal{G}^{u_t,c_t}\tilde{V}(x_t, r_t) - \beta\tilde{V}(x_t, r_t)\right]dt + dm_{1,t} + dm_{2,t}, \tag{5.27}$$

where

$$m_{1,t} \equiv \int_0^t e^{-\beta s}\sigma_1 u_s x_s^\gamma \tilde{W}(r_s) dw_{1,s}, \quad m_{2,t} \equiv \frac{1}{\gamma}\int_0^t e^{-\beta s}\sigma_2 x_s \gamma \tilde{W}_r(r_s) d\tilde{w}_s,$$

and $\tilde{W}(r)$ is defined by $\tilde{W}(r) \equiv e^{\tilde{Z}(r)}$. It is not hard to verify that $\tilde{W}(r)$ is a classical solution of (2.18).

From the definition of $\tilde{W}(r)$, we know that $e^{\underline{Z}(r)} \leq \tilde{W}(r) \leq e^{\bar{Z}(r)}$. Thus, by virtue of the definitions of $\underline{Z}(r)$ and $\bar{Z}(r)$, we can get that $\tilde{W}(r)$ is bounded. In addition, from Lemma 19, we know that $\tilde{Z}_r(r)$ is bounded. Since $\tilde{W}_r(r) = \tilde{W}(r)Z_r(r)$, $\tilde{W}_r(r)$ is also bounded. Therefore, it is not hard to show that $m_{1,t}, m_{2,t}$ are both martingales.

Now integrate (5.27) on $[0,T]$. Since $\tilde{W}(r)$ is a classical solution of (2.18), it is not hard to verify that $\tilde{V}(x,r)$ is a classical solution of (2.14). Then we have

$$e^{-\beta T}\tilde{V}(x_T, r_T) - \tilde{V}(x,r) \leq -\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt + m_{1,T} + m_{2,T}.$$

Take expectation for both sides, and we can get

$$\begin{aligned} \tilde{V}(x,r) &\geq E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt + E\left[e^{-\beta T}\tilde{V}(x_T, r_T)\right] \\ &= E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t x_t)^\gamma dt + \frac{1}{\gamma}E\left[e^{-\beta T}x_T^\gamma \tilde{W}(r_T)\right]. \end{aligned} \tag{5.28}$$

If $J(x,r,u.,c.) = -\infty$, then we must have

$$\tilde{V}(x,r) \geq J(x,r,u.,c.). \tag{5.29}$$

Otherwise, if $J(x,r,u.,c.) > -\infty$, i.e.,

$$\int_0^\infty E\left[e^{-\beta t}c_t^\gamma x_t^\gamma\right] dt < \infty, \tag{5.30}$$

we must have

$$\liminf_{T\to\infty} E\left[e^{-\beta T}c_T^\gamma x_T^\gamma\right] = 0. \tag{5.31}$$

In addition, it not hard to find a constant Λ such that

$$\Lambda[b_1(r-b_2)^2 + b_3] \geq \left[A_1(r-A_2)^2 + A_3\right].$$

Therefore, since $\gamma < 0$, by virtue of (5.21), we can get

$$\begin{aligned} c_t^\gamma &\geq \left[A_1(r_t - A_2)^2 + A_3\right]^\gamma \\ &\geq \Lambda^\gamma\left[b_1(r_t-b_2)^2 + b_3\right]^\gamma \\ &\geq b_3\Lambda^\gamma b_3^{-1}\left[b_1(r_t-b_2)^2 + b_3\right]^\gamma \\ &\geq b_3\Lambda^\gamma\left[b_1(r_t-b_2)^2 + b_3\right]^{\gamma-1} \\ &\geq b_3\Lambda^\gamma\tilde{W}(r_t). \end{aligned}$$

Combined with (5.31), this implies

$$\liminf_{T\to\infty} E\left[e^{-\beta T}x_T^\gamma\tilde{W}(r_T)\right] = 0. \tag{5.32}$$

Then, let $T\to\infty$ in (5.28) and take $\liminf$, and we can get

$$\tilde{V}(x,r) \geq J(x,r,u.,c.). \tag{5.33}$$

On the other hand, for u_t^*, c_t^* defined by (5.26), since $\tilde{Z}_r(r)$ is bounded, it is not hard to verify that

$$0 \le c_t^* \le a_1(r_t - a_2)^2 + a_3, \quad \forall t \ge 0, \tag{5.34}$$

$$E\int_0^T (u_t^*)^2 dt < \infty, \quad \forall T \ge 0. \tag{5.35}$$

So (5.21), (5.22) hold if we take $A_1 \ge a_1, A_2 = a_2$ and $A_3 \ge a_3$. Thus, to ensure that $(u^*, c^*) \in \Pi$, we need to show that (5.23) holds for u_t^*, c_t^*. By Ito's rule, Define $\tilde{l}(r,u) \equiv r + (b-r)u - \frac{1}{2}\sigma_1^2 u^2$. Then using Ito's rule, we can get

$$x_t = x \exp\left\{\int_0^t [\tilde{l}(r_s, u_s^*) - c_s^*]ds + \int_0^t \sigma_1 u_s^* dw_{1,s}\right\}.$$

It is not hard to verify that $e^{\int_0^t [4\gamma\sigma_1 u_s^* dw_{1,s} - 8\gamma^2\sigma_1^2 (u_s^*)^2 ds]}$ is a positive super-martingale which satisfies

$$Ee^{\int_0^t [4\gamma\sigma_1 u_s^* dw_{1,s} - 8\gamma^2\sigma_1^2 (u_s^*)^2 ds]} \le 1.$$

Given above equality and by virtue of the Cauchy-Schwarz Inequality, we can get

$$\begin{aligned}
& E\left[e^{-2\beta t}(u_t^*)^2 x_t^{2\gamma}\right] \\
\le\ & x^{2\gamma} e^{-2\beta t}\left(E\left[(u_t^*)^4 e^{\int_0^t 4\gamma[l(r_s,u_s^*)-c_s^*]ds}\right]\right)^{\frac{1}{2}} \cdot \left(Ee^{\int_0^t [4\gamma\sigma_1 u_s^* dw_{1,s} - 8\gamma^2\sigma_1^2 (u_s^*)^2 ds]}\right)^{\frac{1}{2}} \\
\le\ & x^{2\gamma} e^{-2\beta t}\left(E\left[(u_t^*)^4 e^{\int_0^t 4\gamma[l(r_s,u_s^*)-c_s^*]ds}\right]\right)^{\frac{1}{2}} \\
\le\ & x^{2\gamma} e^{-2\beta t}\left(E[(u_t^*)^8]\right)^{\frac{1}{4}} \cdot \left(Ee^{\int_0^t 8\gamma[l(r_s,u_s^*)-c_s^*]ds}\right)^{\frac{1}{4}}
\end{aligned}$$

where

$$l(r,u) \equiv r + (b-r)u + \left(2\gamma - \frac{1}{2}\right)\sigma_1^2 u^2. \tag{5.36}$$

Since $\tilde{Z}_r$ is bounded, using the Cauchy-Schwarz Inequality, we can get

$$l(r_s, u_s^*) \ge \Lambda_1 + r_s + \frac{(b-r_s)^2}{2\sigma_1^2(1-\gamma)^2}(1 + \frac{5}{2}\gamma). \tag{5.37}$$

In addition, by virtue of (5.34), (5.2), we can get

$$8\gamma[l(r_s,u_s^*) - c_s^*] \le 8\gamma\left[\Lambda_1 + \Lambda_2 r_s + \frac{(1+3\gamma)r_s^2}{2\sigma_1^2(1-\gamma)^2}\right]. \tag{5.38}$$

By Lemma 12, we can get that if (5.24) holds, then

$$Ee^{\int_0^t 8\gamma[l(r_s,u_s^*)-c_s^*]ds} \le \Lambda(T) < \infty, \quad \forall t \in [0,T]. \tag{5.39}$$

In addition, since $\tilde{Z}_r(r)$ is bounded, we can get

$$E\left[(u_t^*)^8\right] \le \Lambda(T) < \infty, \quad \forall t \in [0,T]. \tag{5.40}$$

Therefore, we now have

$$E\left[e^{-2\beta t}(u_t^*)^2 x_t^{2\gamma}\right] \le \Lambda(T), \quad \forall t \in [0,T], \tag{5.41}$$

which implies (5.23). Thus, we have shown that $(u_t^*, c_t^*) \in \Pi$. Given this, instead of (5.28), now we can get

$$\tilde{V}(x,r) = E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt + E\left[e^{-\beta T}\tilde{V}(x_T, r_T)\right]. \qquad (5.42)$$

Since $\tilde{V}(x_T, r_T) \leq 0$, we can get

$$\tilde{V}(x,r) \leq E\int_0^T e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt.$$

Let T goes to $+\infty$, then we have

$$\tilde{V}(x,r) \leq E\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(c_t^* x_t)^\gamma dt,$$

i.e.,

$$\tilde{V}(x,r) \leq J(x, r, u.^*, c.^*). \qquad (5.43)$$

This completes the proof. **Q.E.D.**

References

[BF] Bensoussan, A. and Frehse, J. (2002), *Regularity Results for Nonlinear Elliptic Systems and Applications.* Springer-Verlog, New York.

[BiPl] Bielecki, T. R. and Pliska, S. R. (1997), Risk sensitive dynamic asset management. *Appl. Math. and Optimization* 37, 337-360.

[BSW] Bailey, P. B., Shampine, L. F. and Waltman, P. E. (1968), *Nonlinear Two Point Boundary Value Problems.* Academic Press, New York.

[BS] Bainov, D. and Simeonov, P. (1992), *Integral Inequalities and Applications.* Kluwer Academic Publishers, Norwell, MA.

[Bo] Borkar, V.S. (1989), *Optimal Control of Diffusion Processes.* Logman Scientific and Technical, Essex, UK.

[CIL] Crandall, M. G., Ishii, H., and Lions, P. L. (1992), User's guide to viscosity solutions of second order partial differential equations. *Bulletin of AMS*, Vol. **27**, No. 1, 1-67.

[E] Evans, L. C. (1998), *Partial Differential Equations.* AMS, Providence, RI.

[FiFl] Fitzpatrick, B. G. and Fleming, W. H. (1991), Numerical methods for an optimal investment-consumption model. *Mathematics of Operations Research*, Vol. 16, No. 4, USA.

[Fl] Fleming, W. H. (1995), Optimal investment models and risk-sensitive stochatic control. *IMA Vol. Math. Appl.*, **65**, 35-45, Springer, New York.

[FlHH] Fleming, W. H. and Hernandez-Hernandez, D.(2002), An optimal consumption model with stochastic volatility, to appear in *Finance and Statistics.*

[FlM] Fleming, W. H. and McEneaney, W. M. (1995), Risk-sensitive control on an infinite time horizon. *SIAM J. Control Optim.* **33** 1881-1915.

[FlP] Fleming, W. H. and Pang, T. (2003), A stochastic control model of investment, production and consumption, preprint.

[FlR] Fleming, W. H. and Rishel, R. W. (1975), *Deterministic and Stochastic Optimal Control.* Springer, New York.

[FlSh1] Fleming, W. H. and Sheu, S. J. (1999), Optimal long term growth rate of expected utility of wealth. *Annals of Applied Probability*, Vol. **9**, No. 3, 871-903.

[FlSh2] Fleming, W. H. and Sheu, S. J. (2000), Risk-sensitive control and optimal investment model. *Mathematical Finance*, Vol. **10**, No. 2, 197-213.

[FlSo] Fleming, W. H. and Soner, H. M. (1992), *Controlled Markov Processes and Viscosity Solutions.* Springer, New York.

[FlSt1] Fleming, W. H. and Stein, J. L. (2001), Stochastic inter-temporal optimization in discrete time, in Negishi, Takashi, Rama Ramachandran and Kazuo Mino (ed) Economic Theorey, Dynamics and Markets: Essays in Honor of Ryuzo Sato, Kluwer.

[FlSt2] Fleming, W. H. and Stein, J. L. (2002), Stochastic optimal control in international finance and debt. To appear in *J. of Banking and Finance.*

[Fr] Friedman, A. (1975), *Stochastic Differential Equations and Applications, Vol. 1.* Academic Press, New York.

[GT] Gilbarg, D. and Trudinger, N. S. (1983), *Elliptic Partial Differential Equations of Second Order*, 2nd Edition. Springer, New York.

[KD] Kushner, H. J. and Dupuis, P. (2001), *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd Edition. Springer, New York.

[KS] Karatzas, I. and Shreve, S. E. (1991), *Brownian Motion and Stochastic Calculus*, 2nd Edition. Springer, New York.

[L] Lions, P.-L. (1983), Optimal control of diffusion processes and Hamiltion-Jacobi-Bellman equations, part II: viscosity solutions and uniqueness. *Comm. PDE.*, **8**, 1229-1276.

[LS] Lipster, R. S. and Shiryayev, A. N. (1977), *Statistics of Random Processes I.* Springer, New York.

[M] Merton, R. C. (1992), *Continuous Time Finance*, reveised edition. Blackwell, Cambridge, MA.

[O] Øksendal, B. (1998), *Stochastic Differential Equations*, 5th Edition. Springer, New York.

[Pang] Pang, T. (2002), *Stochastic Control Theory and Its Applications to Fianacial Economics*, Ph.D thesis, Brown University, Providence, RI.

[P] Pao, C. V. (1992), *Nonlinear Parabolic and Elliptic Equations.* Plenum Press, New York.

[UZ] Üstünel, A. S. and Zakai, M. (2000), *Transformation of Measure of Weiner Space.* Springer, New York.

[W] Walter, W. (1998), *Ordinary Differential Equations.* Springer, New York.

[Z] Zariphopoulou, T. (2001), A solution approach to valuation with unhedgeable risks. *Finance and Stochastics*, Vol. 5(1), 61-82.