Portfolio Optimization Models on Infinite-Time Horizon[1]

T. PANG[2]

Oct. 16, 2002 (Last revised Oct. 06, 2003)

Abstract

A portfolio optimization problem on an infinite-time horizon is considered. Risky asset prices obey a logarithmic Brownian motion, and interest rates vary according to an ergodic Markov diffusion process. The goal is to choose optimal investment and consumption policies to maximize the infinite-horizon expected discounted Hyperbolic Absolute Risk Aversion (HARA) utility of consumption. The problem is then reduced to a 1-dimensional stochastic control problem by virtue of the Girsanov transformation. A dynamic programming principle is used to derive the dynamic programming equation (DPE). The subsolution-supersolution method is used to obtain existence of solutions of the DPE. The solutions are then used to derive the optimal investment and consumption policies. In addition, for a special case, the author obtains the results using the viscosity solution method.

Key Words: Portfolio optimization, dynamic programming equations, subsolution and supersolution.

[1]The author thanks Professor Wendell H. Fleming for his many invaluable comments.

[2]Assistant Professor, Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205

1 Introduction

In the classical Merton portfolio optimization problem, an investor can invest his money on a risky asset (e.g. stock) and a riskless asset (e.g. bond). The goal is to maximize the total expected discounted utility of consumption by choosing the optimal investment and consumption strategy. In this model, The interest rate r for the riskless asset is a constant and the risky asset price fluctuates randomly according to a logarithmic Brownian motion. For Hyperbolic Absolute Risk Aversion (HARA) utility function, the Merton problem has a simple explicit solution. The optimal investment strategy is to put the fixed ratio of the total wealth onto the risky asset and the optimal consumption rate is a constant times the wealth. See for example Fleming and Soner (Ref. 1).

However, in our real world, even for the money in the bank, the interest rate may fluctuate from time to time. Therefore, in the present paper the interest rate r_t for the riskless asset is assumed to be an ergodic Markov diffusion process. A typical example is the Vasicek model, in which r_t fluctuates around a certain value at most of the time. Compared to the classical Merton's model, due to the fluctuation of r_t, the optimal investment and consumption policy are no longer constants. Instead, they depend on r_t and the value function $W(r_t)$. See (48) and (69).

Another motivation for this work comes from investment/consumption models in international finance, of a kind considered by Fleming and Stein (Ref. 2). An investment/ consumption model is considered by Fleming and Hernandez-Hernandez (Ref. 3), in which the interest rate is a constant but the volatility of the risky asset price is stochastic. Although the approach in Ref. 3 has some similar features with the present paper, the methods and technical issues to be resolved in the two papers are different.

The dynamic programming method is used for the stochastic control problem considered in the present paper. The state variables are x_t, r_t, where x_t is the wealth. The controls are the fraction u_t of wealth in the risky asset and $c_t = C_t/x_t$ where C_t is the consumption rate. The state dynamics are the stochastic differential equations (7) $-$ (10). HARA utility is considered. The goal is to choose optimal control policies to maximize the objective function defined by (11). The problem is then reduced to a 1-dimensional problem by virtue of the Girsanov transformation. The goal is then to maximize the new defined objective function $\hat{J}$, which is given by (18). This idea is partially from Fleming and Sheu (Ref. 4).

A method of subsolution and supersolution is used to show that the dynamic programming equation (28) has a solution $W(r)$ with appropriate behavior as $|r| \to \infty$. $\gamma > 0$ case and $\gamma < 0$ case are considered in Section 3 and Section 4, respectively. In either case, a pair of subsolution and supersolution of the DPE is obtained to guarantee that a suitable classical solution exists. The solution is then verified to be our value function. The solution can then be used to derive the optimal control policy. These results are given in Theorem 3.1-3.2 and Theorem 4.1-4.2 for $\gamma > 0$ and $\gamma < 0$, respectively. In Section 5, for a special case, the

Vasicek model, viscosity solution method is used to solve the problem. The value function defined by (19) is a viscosity solution of the DPE (22). Then we show that this viscosity is smooth enough, and it is actually a classical solution of (22).

The results in this paper are adapted from the author's PhD Thesis (Ref. 5). In Chapter 2 of Ref. 5, a related optimal investment problem on a finite time horizon $0 \leqslant t \leqslant T$ was also considered. The goal is then to choose an investment control u_t to maximize expected HARA utility of final wealth $E[\gamma^{-1} x_T^{\gamma}]$. This model is of a type previously considered by Bielecki and Pliska (Ref. 6), Zariphopoulou (Ref. 7), Fleming and Sheu (Ref. 4). The analysis for that finite horizon stochastic control problem is considerably simpler than for the optimal investment-consumption model considered in the present paper. In the present paper, it is assumed that the riskless interest rate r_t and the risky asset price P_t are uncorrelated. Unfortunately, the Girsanov transformation technique and the viscosity solution method used in the present paper can not be applied to the correlated case. For correlated case, please refer to Fleming and Pang (Ref. 8).

2 Problem Formulation and Girsanov Transformation

Consider an agent who can invest his money on a risky asset and a riskless asset. The price of the risky asset P_t is modelled by a logarithmic Brownian motion:

$$dP_t = P_t \left[b dt + \sigma_1 dw_{1,t} \right], \tag{1}$$

where b, σ_1 are positive constants, and $w_{1,t}$ is a standard Brownian motion. The interest rate of the riskless asset may change from time to time. Here we use a stochastic process to describe its behavior:

$$dr_t = f(r_t) dt + \sigma_2 dw_{2,t}, \tag{2}$$

where $\sigma_2 > 0$ is a constant, and $w_{2,t}$ is a standard 1-dimensional Brownian motion which is independent of $w_{1,t}$. We will consider the generalized Vasicek model:

$$f(r) \in \mathbf{C}^2(\mathbf{R}), \tag{3}$$

$$|f_{rr}(r)| \leqslant K(1 + |r|^{\alpha}), \tag{4}$$

$$-c_2 \leqslant f_r(r) \leqslant -c_1, \tag{5}$$

where σ_2, K, α, c_1 and c_2 are positive constants. One example is the Vasicek model:

$$dr_t = -\bar{c}(r_t - \bar{r}) dt + \sigma_2 dw_{2,t}, \tag{6}$$

where $\bar{c}, \bar{r}$ are constants. In this model, r_t will spend most of its time around $\bar{r}$.

Denote x_t as the total wealth of the agent at time t. Here we require that $x_t > 0$ all the time. Suppose at time t, a fraction u_t of his money will be invested on the risky asset, and his consumption rate is C_t. Thus, a fraction of $(1 - u_t)$

will be invested on the riskless asset. For technique reasons, we use $c_t \equiv C_t/x_t$ in stead of C_t. Suppose the initial wealth is x and the initial interest rate is r. Then we can write the SDEs of (x_t, r_t) as

$$\begin{aligned} dx_t &= x_t[r_t + (b - r_t)u_t - c_t]dt + \sigma_1 u_t x_t dw_{1,t}, && (7) \\ x_0 &= x, && (8) \\ dr_t &= f(r_t)dt + \sigma_2 dw_{2,t}, && (9) \\ r_0 &= r. && (10) \end{aligned}$$

We consider the utility functions of the HARA type:

$$U(C) = \frac{1}{\gamma}c^{\gamma}, \quad -\infty < \gamma < 1, \ \gamma \neq 0.$$

Therefore, our goal is to maximize the objective function

$$J(x, r, u., c.) \equiv E_{x,r} \int_0^{\infty} e^{-\beta t} \frac{1}{\gamma}(c_t x_t)^{\gamma} dt. \tag{11}$$

Next, we rewrite the expectation $E(c_t x_t)^{\gamma}$ in terms of an expected exponential-of-integral criterion, which involves only u_t, c_t and r_t. We make precise below the class of admissible controls u_t, c_t, and justify the steps in the following derivation.

According to the equation of x_t, using the Ito's rule, we can get

$$d\log x_t = \left[r_t + (b - r_t)u_t - c_t - \frac{1}{2}\sigma_1^2 u_t^2\right] dt + \sigma_1 u_t dw_{1,t}.$$

Then we can get

$$x_t = x \exp\left\{\int_0^t [\tilde{l}(r_s, u_s) - c_s]ds + \int_0^t \sigma_1 u_s dw_{1,s}\right\},$$

where

$$\tilde{l}(r, u) \equiv r + (b - r)u - \frac{1}{2}\sigma_1^2 u^2.$$

Therefore, we have

$$E_{x,r}(c_t x_t)^{\gamma} = x^{\gamma} E_r c_t^{\gamma} \exp\left\{\int_0^t \gamma[\tilde{l}(r_s, u_s) - c_s]ds + \gamma \int_0^t \sigma_1 u_s dw_{1,s}\right\}.$$

To get rid of the stochastic integral term, we write

$$\gamma \int_0^t \sigma_1 u_s dw_{1,s} = \int_0^t \left[\gamma\sigma_1 u_s dw_{1,s} - \frac{1}{2}\gamma^2\sigma_1^2 u_s^2 ds\right] + \frac{1}{2}\int_0^t \gamma^2\sigma_1^2 u_s^2 ds.$$

In addition, assume

$$M_t = \exp\left\{\int_0^t \gamma\sigma_1 u_s dw_{1,s} - \frac{1}{2}\int_0^t \gamma^2\sigma_1^2 u_s^2 ds\right\}. \tag{12}$$

Then if we have

$$EM_t = 1, \quad \forall t > 0, \tag{13}$$

we can define a new probability measure $\hat{P}$ as

$$d\hat{P} = M_t dP.$$

Under the new probability measure, we have

$$E(c_t x_t)^\gamma = x^\gamma \hat{E} c_t^\gamma \exp\left\{\int_0^t \gamma[l(r_s, u_s) - c_s]ds\right\}, \tag{14}$$

where

$$l(r, u) = r + (b - r)u - \frac{1}{2}(1 - \gamma)\sigma_1^2 u^2. \tag{15}$$

Noting (11) and (14), we know intuitively that to maximize $J(x, r, u., c.)$, it is sufficient to maximize

$$\frac{1}{\gamma}\int_0^\infty e^{-\beta t}\hat{E}_r c_t^\gamma \exp\left\{\int_0^t \gamma[l(r_s, u_s) - c_s]ds\right\} dt.$$

Since $w_{1,t}$ and $w_{2,t}$ are independent, the stochastic equation of r_t is still (9). We regard (9) as the dynamics of a stochastic control problem, in which r_t is the state, and (u_t, c_t) is the control at time t. The newly defined objective function $\hat{J}$ and the value function $\hat{W}(r)$ will be given in (18) and (19).

We require that the control $(u_t, c_t) \in \mathbf{R}^2$ for any $t \geqslant 0$, and it is $\mathcal{F}_{2,t}$-progressively measurable, where $(\mathcal{F}_{2,t}, t \geqslant 0)$ is an increasing family of σ-algebras generated by $(w_{2,t}, t \geqslant 0)$. See Fleming and Soner (Ref. 1 Chapter 4) for details. In some cases, (u_t, c_t) may be obtained from locally Lipschitz continuous control policies $(\underline{\mathrm{u}}, \underline{\mathrm{c}})$:

$$u_t = \underline{\mathrm{u}}(t, r_t), \;\; c_t = \underline{\mathrm{c}}(t, r_t).$$

In addition, we assume that $c_t \geqslant 0$, and there is no constraints for the value of u_t. We also require that

$$P\left\{\int_0^T u_t^2 dt < \infty\right\} = 1, \quad \forall T > 0. \tag{16}$$

Thus all the stochastic integrals we used above are well-defined.

Since $w_{1,t}$ and $w_{2,t}$ are independent, $\mathcal{F}_{2,t}$ and any σ-algebra generated by $w_{1,t}$ are also independent. Noting that u_t is $\mathcal{F}_{2,t}$-progressively measurable, we can get

$$E\exp\left\{\int_0^t [\gamma\sigma_1 u_s dw_{1,s} - \frac{1}{2}\gamma^2\sigma_1^2 u_s^2 ds]\right\} = 1. \tag{17}$$

That is, (13) holds. So the Girsanov transformation is justified. For details to prove (17), please see Üstünel and Zakai (Ref. 9, page 27).

The admissible control spaces will be specified later in Section 3 and Section 4 in case $\gamma > 0$ and $\gamma < 0$, respectively. Conditions to ensure that the value function is finite will also be given.

Now, our goal to choose (u_t, c_t) to maximize the objective function

$$\hat{J}(r, u., c.) \equiv \frac{1}{\gamma}\int_0^\infty e^{-\beta t}\hat{E}_r c_t^\gamma \exp\left\{\int_0^t \gamma[l(r_s, u_s) - c_s]ds\right\} dt. \tag{18}$$

Thus, the value function is

$$\hat{W}(r) = \sup_{u.,c.} \hat{J}(r, u., c.). \tag{19}$$

Remark 2.1 Since $w_{1,t}$ and $w_{2,t}$ are independent, the Girsanov transformation does not change the law of r_t under the new probability measure. Therefore, $\hat{E}_r$ and E_r will be the same. For notation reasons, we will use E_r instead of $\hat{E}_r$ in the following parts of this paper.

Using the dynamic programming principle, we can write the DPE for $\hat{W}(r)$ as

$$\begin{aligned} \beta\hat{W} &= \sup_u\left[\gamma(b-r)u\hat{W} + \frac{1}{2}\gamma(\gamma-1)\sigma_1^2u^2\hat{W}\right] + \gamma r\hat{W} \\ &\quad + f(r)\hat{W}_r + \frac{1}{2}\sigma_2^2\hat{W}_{rr} + \sup_{c\geqslant 0}\left[-\gamma c\hat{W} + \frac{1}{\gamma}c^\gamma\right]. \end{aligned} \tag{20}$$

The potential optimal control policy is

$$u^*(r_t) = \frac{(b-r_t)}{(1-\gamma)\sigma_1^2}, \quad c^*(r_t) = [\gamma\hat{W}(r_t)]^{\frac{1}{\gamma-1}}. \tag{21}$$

Therefore, the differential equation for $\hat{W}$ now is

$$\begin{aligned} \beta\hat{W} &= \frac{\gamma(b-r)^2\hat{W}}{2(1-\gamma)\sigma_1^2} + \gamma r\hat{W} \\ &\quad + f(r)\hat{W}_r + \frac{1}{2}\sigma_2^2\hat{W}_{rr} + \frac{1}{\gamma}(1-\gamma)[\gamma\hat{W}]^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{22}$$

Define

$$W(r) \equiv \gamma\hat{W}(r). \tag{23}$$

Then the equation for $W(r)$ can be written as

$$\frac{1}{2}\sigma_2^2W_{rr} + f(r)W_r + [\gamma Q(r) - \beta]W + (1-\gamma)W^{\frac{\gamma}{\gamma-1}} = 0, \tag{24}$$

where

$$Q(r) = \frac{(b-r)^2}{2(1-\gamma)\sigma_1^2} + r. \tag{25}$$

Define

$$\begin{aligned} \hat{L}W &= \frac{1}{2}\sigma_2^2W_{rr} + f(r)W_r, \qquad (26) \\ \hat{h}(r, W) &= (\gamma Q(r) - \beta)W + (1-\gamma)W^{\frac{\gamma}{\gamma-1}}. \qquad (27) \end{aligned}$$

Then the equation of W can be written as

$$-\hat{L}W = \hat{h}(r, W). \tag{28}$$

Actually, $\hat{L}$ is uniformly elliptic. Now let us define subsolutions and supersolutions.

Definition 2.1 A function $\underline{W}$ is said to be a subsolution of (28) if

$$-\hat{L}\underline{W} \leqslant \hat{h}(r, \underline{W}). \tag{29}$$

$\bar{W}$ is a supersolution if

$$-\hat{L}\bar{W} \geqslant \hat{h}(r, \bar{W}). \tag{30}$$

In addition, if

$$\underline{W}(r) \leqslant \bar{W}(r), \quad \forall r \in \mathbf{R}, \tag{31}$$

then $\langle \underline{W}, \bar{W} \rangle$ is a pair of ordered subsolution and supersolution of (28).

In the next two sections, we will find a pair of subsolution and supersolution of (28) and prove that it has a classical $\mathbf{C}^2$ solution. This will be done for $\gamma > 0$ and $\gamma < 0$ in Section 3 and Section 4, respectively, using the subsolution-supersolution method. The solution is then verified to be equal to our value function, in both cases.

3 $\gamma > 0$ Case

In this section, we obtain a pair of ordered subsolution and supersolution for equation (28). Then we use the subsolution-supersolution method to get a classical solution $\tilde{W}(r)$. The function $\frac{1}{\gamma}\tilde{W}(r)$ will be verified to be equal to our value function $\hat{W}(r)$ which is defined in (19), and the solution will then be used to derive the optimal control policy.

First, let us get a subsolution of (28).

Lemma 3.1 Suppose $\gamma > 0$ and

$$\beta > \gamma b - \frac{\sigma_1^2}{2}\gamma(1-\gamma). \tag{32}$$

Let K_1 be a positive constant given by

$$K_1^{\frac{1}{\gamma-1}} = \frac{1}{1-\gamma}\left[\beta - b\gamma + \frac{\sigma_1^2}{2}\gamma(1-\gamma)\right]. \tag{33}$$

Then, any constant $K_2 \in [0, K_1]$ is a subsolution of (28).

The proof is rather straight, so it is omitted here.

To get a pair of ordered subsolution and supersolution, we also need a supersolution of (28). The following lemma is needed.

Lemma 3.2 Define

$$\gamma_1 \equiv \frac{\sigma_1^2 c_1^2}{\sigma_1^2 c_1^2 + \sigma_2^2}, \tag{34}$$

and assume that

$$0 < \gamma < \gamma_1. \tag{35}$$

Then the equation

$$-2\sigma_2^2 a^2 + 2c_1 a - \frac{\gamma}{2\sigma_1^2(1-\gamma)} = 0. \tag{36}$$

has two different real, positive roots.

The result is easy to be verified by virtue of direct calculation. Suppose a^-, a^+ are the roots of (36) which satisfy $0 < a^- < a^+$. Define $I_1 \equiv (a^-, a^+)$. Let $a_1 > 0$ be a constant such that $a_1 \in I_1$. Define

$$\mu_1(a_1) \equiv -2\sigma_2^2 a_1^2 + 2c_1 a_1 - \frac{\gamma}{2\sigma_1^2(1-\gamma)}, \tag{37}$$

$$\mu_2(a_1) \equiv -2a_1 f(0) - \gamma + \frac{b\gamma}{\sigma_1^2(1-\gamma)}, \tag{38}$$

$$\mu_3(a_1) \equiv -a_1\sigma_2^2 - \frac{\gamma b^2}{2\sigma_1^2(1-\gamma)}, \tag{39}$$

$$C_1(a_1) \equiv \frac{4\mu_1(a_1)\mu_3(a_1) - \mu_2(a_1)^2}{4\mu_1(a_1)}. \tag{40}$$

Then we have the following lemma.

Lemma 3.3 Suppose (35) holds and K_1 is the constant given by (33). Then there exists a constant $A > K_1$ such that

$$\bar{W}(r) \equiv Ae^{a_1 r^2} \tag{41}$$

is a supersolution of (28) provided that

$$\beta > -C_1(a_1), \tag{42}$$

where $C_1(a_1)$ is defined by (40).

Proof. It is easy to verify that

$$\bar{W}_r = 2a_1 r\bar{W}, \quad \bar{W}_{rr} = (2a_1 + 4a_1^2 r^2)\bar{W}.$$

Then we have

$$-\hat{L}\bar{W} = [-2a_1^2\sigma_2^2 r^2 - 2a_1 r f(r) - a_1\sigma_2^2]\bar{W}.$$

By virtue of (5) and the mean-value theorem, we can get

$$-2a_1 r f(r) = -2a_1 r[f(0) + f_r(\xi)r] \geqslant 2c_1 a_1 r^2 - 2a_1 f(0) r,$$

where $\xi \in [0, r]$. Therefore, we have

$$-\hat{L}\bar{W} \geqslant [(2c_1a_1 - 2a_1^2\sigma_2^2)r^2 - 2a_1f(0)r - a_1\sigma_2^2]\bar{W}.$$

Since $\bar{W} > 0$, by the definition of $\hat{h}$, to show that $\bar{W}$ defined by (41) is a supersolution of (28), it is sufficient to show that

$$\mu_1(a_1)r^2 + \mu_2(a_1)r + \mu_3(a_1) \geqslant -\beta + (1-\gamma)A^{\frac{1}{\gamma-1}}e^{\frac{a_1r^2}{\gamma-1}}, \tag{43}$$

where $\mu_i(a_1), i = 1, 2, 3$ are given be (37) – (39). Direct calculation implies that $\mu_1(a_1) > 0$. Then it is not hard to verify that the left hand side of (43) is bounded below by $C_1(a_1)$. Since $0 < \gamma < \gamma_1 < 1$, we can get that $\exp\left\{a_1r^2/(\gamma-1)\right\} \leqslant 1$. Thus, if (42) holds, then we can take A large enough such that

$$(1-\gamma)A^{\frac{1}{\gamma-1}}e^{\frac{a_1r^2}{\gamma-1}} \leqslant (1-\gamma)A^{\frac{1}{\gamma-1}} \leqslant \beta + C_1(a_1),$$

which implies (43). □

Given a pair of subsolution and supersolution, now we can get a classical solution of (28).

Theorem 3.1 Suppose (32), (35) and (42) hold. Then (28) has a classical solution $\tilde{W}$ such that

$$K_1 \leqslant \tilde{W}(r) \leqslant \bar{W}(r), \tag{44}$$

where K_1 and $\bar{W}(r)$ are defined by (33) and (41), respectively.

Proof. By the definitions of K_1 and $\bar{W}(r)$, they are ordered subsolution and supersolution of (28). Now by virtue of Pao (Ref. 10 Theorem 7.5.2, page 322, where subsolutions and supersolutions are referred as lower-solutions and upper-solutions), we can get the result. □

Denote $(\mathcal{F}_{2,t}, t \geqslant 0)$ the family of σ-algebras generated by $(w_{2,t}, t \geqslant 0)$. We have the following admissible control space.

Definition 3.1 (Admissible Control Space) A control (u_t, c_t) is said to be in the admissible control space Π if it satisfies:

(i.) (u_t, c_t) is $\mathcal{F}_{2,t}$ -progressively measurable;

(ii.) (16) holds for u_t;

(iii.) $c_t \geqslant 0$.

Next we will verify that $\tilde{W}(r) \equiv \gamma\hat{W}(r)$. This will be done in Theorem 3.2. Actually, we need a condition stronger than (35). Define

$$\bar{\gamma} \equiv \frac{7\sigma_1^2c_1^2}{7\sigma_1^2c_1^2 + 16\sigma_2^2}. \tag{45}$$

we require that

$$0 < \gamma < \bar{\gamma}. \tag{46}$$

Theorem 3.2 Let $0 < \gamma < \bar{\gamma}$, where $\bar{\gamma}$ is given by (45). Suppose (32), (42) holds. Let $\tilde{W}(r)$ be the classical solution of (28) which satisfies (44). Then, if β is suitable large, we have

$$\tilde{W}(r) \equiv \gamma \hat{W}(r). \tag{47}$$

In addition, $\hat{J}(r, u., c.)$ reaches its maximum at

$$u^*(r_t) = \frac{(b - r_t)}{(1-\gamma)\sigma_1^2}, \quad c^*(r_t) = [\tilde{W}(r_t)]^{\frac{1}{\gamma-1}}. \tag{48}$$

The proof is rather standard, and it involves a lot of stochastic calculations. Please see Pang (Ref. 5 Theorem 1.6) for details.

4 $\gamma < 0$ Case

Similarly to last section, we will find a pair of subsolution and supersolution of the equation (28) in case $\gamma < 0$. Then we use the subsolution-supersolution method to get a classical solution $\tilde{W}(r)$. $\frac{1}{\gamma}\tilde{W}(r)$ will be verified to equal our value function $\hat{W}(r)$ defined in (19), and the solution is then used to derive the optimal control policy.

Unlike $\gamma > 0$ case, equation (28) now has a constant supersolution, instead of a constant subsolution.

Lemma 4.1 Suppose $\gamma < 0$ and

$$\beta > b\gamma - \frac{1}{2}\sigma_1^2\gamma(1-\gamma). \tag{49}$$

Let K_1 be a positive constant given by

$$K_1^{\frac{1}{\gamma-1}} = \frac{1}{1-\gamma}\left[\beta - b\gamma + \frac{\sigma_1^2}{2}\gamma(1-\gamma)\right]. \tag{50}$$

Then, any constant $K_2 > K_1$ is a supersolution of (28).

The results are not hard to verify by virtue of the definition of supersolutions and the proof is omitted here. Next we will get a positive subsolution of (28).

Lemma 4.2 Define

$$\tilde{a}_1 \equiv \frac{1}{2\sigma_1^2(1-\gamma)}, \quad a_2 \equiv b - \sigma_1^2(1-\gamma). \tag{51}$$

Assume that a_1 and $a_3 > 0$ satisfy

$$0 < a_1 < \tilde{a}_1, \tag{52}$$

$$(1-\gamma)a_3 + b\gamma - \frac{1}{2}\sigma_1^2\gamma(1-\gamma) - \beta \geqslant (1-\gamma)\left[\frac{a_1}{a_3}\sigma_2^2 + |f(a_2)|\sqrt{\frac{a_1}{a_3}}\,\right]. \tag{53}$$

Define

$$\underline{W}(r) \equiv [a_1(r-a_2)^2 + a_3]^{\gamma-1}. \tag{54}$$

Then $\underline{W}$ is a subsolution of (28).

The proof is rather direct. Please see Pang (Ref. 5 Lemma 1.30) for details. The following lemma is needed in the verification theorem.

Lemma 4.3 Define $\tilde{a}_3$ such that

$$\tilde{a}_3 \equiv (1-\gamma)\left[\frac{\tilde{a}_1}{\tilde{a}_3}\sigma_2^2 + |f(a_2)|\sqrt{\frac{\tilde{a}_1}{\tilde{a}_3}}\right], \tag{55}$$

where $\tilde{a}_1$ and a_2 are given by (51). If

$$\beta > -\gamma\tilde{a}_3 + b\gamma - \frac{1}{2}\sigma_1^2\gamma(1-\gamma), \tag{56}$$

then there exist $a_1 > 0, a_3 > 0$ such that $W_0^{\gamma-1}$ is a subsolution of (28), where

$$W_0(r) \equiv a_1(r-a_2)^2 + a_3, \tag{57}$$

and a_2 is given by (51). In addition, for $Q(r)$ defined by (25), there exists $\delta > 0$ such that

$$\gamma Q(r) - \gamma W_0(r) - \beta \leqslant -\delta. \tag{58}$$

Proof. By definition, it is not hard to verify that $\frac{\gamma}{2\sigma_1^2(1-\gamma)} - \gamma\tilde{a}_1 = 0$. Since $\tilde{a}_1 > 0$, there exists an $\epsilon_1 > 0$ such that

$$0 < a_1 \equiv \tilde{a}_1 - \epsilon_1 < \tilde{a}_1. \tag{59}$$

Thus, we must have

$$\frac{\gamma}{2\sigma_1^2(1-\gamma)} - \gamma a_1 < 0. \tag{60}$$

Given this, noting (25) and (57), we can get

$$\gamma Q(r) - \gamma W_0(r) - \beta \leqslant -\left[\beta + \gamma a_3 - b\gamma + \frac{\sigma_1^2}{2}\gamma(1-\gamma)\right]. \tag{61}$$

Define $\delta_1 \equiv \beta + \gamma\tilde{a}_3 - b\gamma + \frac{1}{2}\sigma_1^2\gamma(1-\gamma)$. By virtue of (56), we have that $\delta_1 > 0$. So we can take a constant δ such that

$$0 < \delta < \min\left\{\delta_1, (1-\gamma)\frac{\epsilon_1}{\tilde{a}_3}\sigma_2^2\right\}. \tag{62}$$

Since $\delta < \delta_1$, noting that $\gamma < 0$, we can take $a_3 > \tilde{a}_3$ such that

$$\beta + \gamma a_3 - b\gamma + \frac{1}{2}\sigma_1^2\gamma(1-\gamma) = \delta > 0. \tag{63}$$

Now by virtue of (61), we can get (58).

In addition, since $a_3 > \tilde{a}_3$, using (63), (55), (59) and (62), we can get

$$\begin{aligned}
& (1-\gamma)a_3 + b\gamma - \frac{1}{2}\sigma_1^2\gamma(1-\gamma) - \beta \\
= \;& a_3 - \delta \\
> \;& (1-\gamma)\left[\frac{\tilde{a}_1}{\tilde{a}_3}\sigma_2^2 + |f(a_2)|\sqrt{\frac{\tilde{a}_1}{\tilde{a}_3}}\right] - \delta \\
\geqslant \;& (1-\gamma)\left[\frac{a_1}{\tilde{a}_3}\sigma_2^2 + |f(a_2)|\sqrt{\frac{a_1}{\tilde{a}_3}}\right] + (1-\gamma)\frac{\epsilon_1}{\tilde{a}_3}\sigma_2^2 - \delta \\
\geqslant \;& (1-\gamma)\left[\frac{a_1}{\tilde{a}_3}\sigma_2^2 + |f(a_2)|\sqrt{\frac{a_1}{\tilde{a}_3}}\right] \\
\geqslant \;& (1-\gamma)\left[\frac{a_1}{a_3}\sigma_2^2 + |f(a_2)|\sqrt{\frac{a_1}{a_3}}\right].
\end{aligned}$$

Combined with (61), this implies (53). Then, by Lemma 4.2, $W_0^{\gamma-1}$ is a subsolution of (28). □

Theorem 4.1 Suppose $\gamma < 0$ and (56) holds. Define $\underline{W}(r)$ as in (54). Suppose K_2 is a constant such that

$$K_2 \geqslant K_1, \quad K_2 \geqslant \underline{W}(r), \ \forall r. \tag{64}$$

Then, the equation (28) has a classical solution $\tilde{W}(r)$ which satisfies

$$\underline{W}(r) \leqslant \tilde{W}(r) \leqslant K_2. \tag{65}$$

Proof. Suppose $W(r)$ satisfies (28). Define $Z(r) \equiv K_2 - W(r)$. Then the equation for $Z(r)$ is

$$-\hat{L}Z = -(\gamma Q(r) - \beta)[K_2 - Z(r)] - (1-\gamma)[K_2 - Z(r)]^{\frac{\gamma}{\gamma-1}}. \tag{66}$$

Since $\gamma < 0$ and $\tilde{a}_3 > 0$, (56) implied (49). Therefore, by Lemma 4.1, any $K_2 > K_1$ is a supersolution of (28). In addition, by Lemma 4.2, $\underline{W}(r)$ is a subsolution of (28). It is not hard to check that 0 and $K_2 - \underline{W}(r)$ are (66)'s subsolution and supersolution, respectively.

Now according to Pao (Ref. 10 Theorem 7.5.2, page 322, where subsolutions and supersolutions are referred as lower-solutions and upper-solutions.), equation (66) has a classical solution $\tilde{Z}(r)$ such that $0 < \tilde{Z}(r) \leqslant K_2 - \underline{W}(r)$. Define $\tilde{W}(r) \equiv K_2 - \tilde{Z}(r)$. Then it is easy to verify that $\tilde{W}(r)$ is a classical solution of (28) and it satisfies (65). □

Next we need a verification theorem to make sure that $\tilde{W}(r)$ is the solution we need. Before we do that, let us define the admissible control space Π. Denote $(\mathcal{F}_{2,t}, t \geqslant 0)$ the family of σ-algebras generated by $(w_{2,t}, t \geqslant 0)$. The admissible control space is defined as the following.

Definition 4.1 (Admissible Control Space) A control (u_t, c_t) is said to be in the admissible control space Π if it satisfies:

(i.) (u_t, c_t) is $\mathcal{F}_{2,t}$ -progressively measurable;

(ii.) (16) holds for u_t;

(iii.) $c_t \geqslant 0$;

(iv.) For $Q(r)$ defined by (25), there exists a $\delta > 0$ such that

$$0 \leqslant c_t \leqslant Q(r_t) - \frac{\beta}{\gamma} - \delta. \tag{67}$$

Then we have the following verification theorem:

Theorem 4.2 Let $\gamma < 0$. Define $\hat{W}(r)$ as in (19) and suppose (56) holds. Suppose $\tilde{W}(r)$ is a classical solution of (28) which satisfies (65). Then we have

$$\tilde{W}(r) \equiv \gamma \hat{W}(r). \tag{68}$$

In addition, if we define

$$u_t^* \equiv \frac{(b - r_t)}{(1-\gamma)\sigma_1^2}, \quad c_t^* \equiv [\tilde{W}(r_t)]^{\frac{1}{\gamma-1}}, \tag{69}$$

then $(u_t^*, c_t^*) \in \Pi$, and $\hat{J}(r, u., c.)$ reaches its maximum at (u_t^*, c_t^*).

The proof is rather standard. Please see Pang (Ref. 5 Theorem 1.7) for details.

5 Viscosity Solution Method for the Vasicek Model

In this part we will suppose that $\gamma > 0$. Consider the Vasicek Model:

$$f(r) = -c_1(r - \bar{r}), \tag{70}$$

where $c_1 > 0$ and $\bar{r}$ are constants. Although this case is included in the models we studied in Section 3, we will use a different method–viscosity solution method–to get the results we expected. Put differently, we will use the viscosity solution method to get the result that the value function $\hat{W}(r)$ which is defined by (19) is a classical solution of the dynamic programming equation (75). For definitions of viscosity solutions, please refer to Fleming and Soner (Ref. 1 page 64 Section II.4).

The SDEs of (x_t, r_t) are

$$\begin{aligned} dx_t &= x_t[r_t + (b - r_t)u_t - c_t]dt + \sigma_1 u_t x_t dw_{1,t}, & (71)\\ x_0 &= x, & (72)\\ dr_t &= -c_1(r_t - \bar{r})dt + \sigma_2 dw_{2,t}, & (73)\\ r_0 &= r, & (74) \end{aligned}$$

where $w_t = (w_{1,t}, w_{2,t})'$ is a standard 2-dimensional Brownian motion, and $x > 0$ and r are the initial wealth and the initial interest rate, respectively.

Denote $(\mathcal{F}_{2,t}, t \geqslant 0)$ the family of σ-algebras generated by $(w_{2,t}, t \geqslant 0)$. The admissible control space is:

Definition 5.1 (Admissible Control Space) A control (u_t, c_t) is said to be in the admissible control space Π if it satisfies:

(i.) (u_t, c_t) is $\mathcal{F}_{2,t}$ -progressively measurable;

(ii.) (16) holds for u_t;

(iii.) $c_t \geqslant 0$, and c_t is bounded.

The objective function $\hat{J}$ and the value function $\hat{W}$ are defined by (18) and (19). Using the dynamic programming principle, we can write the DPE for $\hat{W}(r)$ as

$$\begin{aligned} \beta\hat{W} &= \sup_u \left[\gamma(b-r)u\hat{W} + \frac{1}{2}\gamma(\gamma-1)\sigma_1^2 u^2 \hat{W}\right] + \gamma r \hat{W} \\ &\quad - c_1(r-\bar{r})\hat{W}_r + \frac{1}{2}\sigma_2^2 \hat{W}_{rr} + \sup_{c \geqslant 0}\left[-\gamma c \hat{W} + \frac{1}{\gamma} c^\gamma\right]. \end{aligned} \tag{75}$$

The potential optimal control policy is

$$u^*(r_t) = \frac{(b-r_t)}{(1-\gamma)\sigma_1^2}, \quad c^*(r_t) = [\gamma \hat{W}(r_t)]^{\frac{1}{\gamma-1}}. \tag{76}$$

Therefore, the differential equation for $\hat{W}$ now is

$$\begin{aligned} \beta\hat{W} &= \frac{\gamma(b-r)^2\hat{W}}{2(1-\gamma)\sigma_1^2} + \gamma r \hat{W} \\ &\quad - c_1(r-\bar{r})\hat{W}_r + \frac{1}{2}\sigma_2^2\hat{W}_{rr} + \frac{1}{\gamma}(1-\gamma)[\gamma\hat{W}]^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{77}$$

Before we go further, we need some lemmas. In those lemmas, we always assume that $(r_t, t \geqslant 0)$ is a solution of (73) – (74).

Lemma 5.1 Suppose $v(r) \in \mathbf{C}^2(\mathbf{R})$ is bounded. In addition, suppose v_r, v_{rr} are all bounded. Define $\phi(r,T) \equiv E_r \exp\left\{\int_0^T v(r_t)dt\right\}$. Then $\phi(r,T) \in \mathbf{C}^{2,1}(\mathbf{R}, [0,\infty))$ and it is a classical solution of

$$\begin{cases} \phi_T = \frac{1}{2}\sigma_2^2\phi_{rr} - c_1(r-\bar{r})\phi_r + v(r)\phi, \\ \phi(r,0) = 1. \end{cases} \tag{78}$$

The proof is rather standard. See Pang (Ref. 5 Lemma 1.12) for details.

Lemma 5.2 Suppose $\hat{v}(r) \in \mathbf{C}^2(\mathbf{R})$. In addition, suppose $\hat{v}, \hat{v}_r, \hat{v}_{rr}$ are all bounded. Define $\eta(r,T) \equiv E_r \exp\{\hat{v}(r_T)\}$. Then $\eta(r,T) \in \mathbf{C}^{2,1}(\mathbf{R}, [0,\infty))$ and it is a classical solution of

$$\begin{cases} \eta_T = \frac{1}{2}\sigma_2^2\eta_{rr} - c_1(r-\bar{r})\eta_r, \\ \eta(r,0) = e^{\hat{v}(r)}. \end{cases} \tag{79}$$

Proof. This is a direct corollary of Theorem 5.6.1 of Friedman (Ref. 11). □

Next we will use the viscosity solution method to get the result that the value function $\hat{W}(r)$ is a classical solution of (75).

First we will show that $\hat{W}(r) < \infty$ under certain conditions. Define

$$\Phi(r,t) = \sup_u \hat{E}_r \exp\left\{\int_0^t \gamma l(r_s, u_s)ds\right\}, \tag{80}$$

where $l(r,u)$ is defined by (15). Noting that $l(r,u)$ is quadratic with respect to u, since r_t does not depend on the value of u_t, it is easy to verify that

$$\Phi(r,t) = \hat{E}_r \exp\left\{\int_0^t \gamma Q(r_s)ds\right\},$$

where $Q(r) = \frac{(b-r)^2}{2(1-\gamma)\sigma_1^2} + r$ as in (25).

Define

$$\gamma_1 \equiv \frac{\sigma_1^2 c_1^2}{\sigma_1^2 c_1^2 + \sigma_2^2}, \tag{81}$$

and assume that

$$0 < \gamma < \gamma_1. \tag{82}$$

Then according to Lemma 3.2, the equation

$$-2\sigma_2^2 a^2 + 2c_1 a - \frac{\gamma}{2\sigma_1^2(1-\gamma)} = 0. \tag{83}$$

has two different real, positive roots. Suppose a^-, a^+ are the two roots which satisfy $0 < a^- < a^+$. Let a be a constant such that

$$a^- < a < a^+. \tag{84}$$

Define

$$\nu_1(a) \equiv -2\sigma_2^2 a^2 + 2c_1 a - \frac{\gamma}{2\sigma_1^2(1-\gamma)}, \tag{85}$$

$$\nu_2(a) \equiv -2c_1\bar{r}a - \gamma + \frac{b\gamma}{\sigma_1^2(1-\gamma)}, \tag{86}$$

$$\nu_3(a) \equiv -\sigma_2^2 a - \frac{\gamma b^2}{2\sigma_1^2(1-\gamma)}, \tag{87}$$

$$C_2(a) \equiv \frac{4\nu_1(a)\nu_3(a) - \nu_2(a)^2}{4\nu_1(a)}. \tag{88}$$

Now let us get a bound for $\Phi(r,t)$.

Lemma 5.3 Suppose (82), (84) hold. Define $C_2(a)$ as in (88). Then for any

$$\beta_1 > -C_2(a), \tag{89}$$

there is a constant A, which is independent of T, such that

$$e^{-\beta_1 T}\Phi(r,T) \leqslant Ae^{ar^2}. \tag{90}$$

Proof. Define a sequence of functions $\{Q_M(r), M = 1, 2, 3, ...\}$ such that

$$Q_M(r) \in \mathbf{C}^\infty; \quad |Q_M(r)| \leqslant M; \quad \left|\frac{\partial Q_M(r)}{\partial r}\right| \leqslant \hat{M}, \quad \left|\frac{\partial^2 Q_M(r)}{\partial r^2}\right| \leqslant \hat{M};$$
$$Q_{M_1}(r) \leqslant Q_{M_2}(r) \leqslant Q(r), \ \forall M_1 < M_2; \quad \lim_{M\to\infty} Q_M(r) = Q(r),$$

where $\hat{M}$ is a constant that may depend on M. Define

$$\psi(r, T) \equiv e^{-\beta_1 T} E_r e^{\int_0^T \gamma Q_M(r_t)dt},$$

then $\psi(r, T) \in \mathbf{C}^{2,1}(\mathbf{R}, [0, \infty))$, and it is a solution of the problem

$$\begin{cases} \frac{\partial \psi}{\partial T} = \frac{\sigma_2^2}{2}\frac{\partial^2 \psi}{\partial r^2} - c_1(r - \bar{r})\frac{\partial \psi}{\partial r} + [\gamma Q_M(r) - \beta_1]\psi, \\ \psi(r, 0) = 1. \end{cases} \tag{91}$$

Define $\bar{\psi}(r) \equiv Ae^{ar^2}$. Then for A large enough, it is not hard to verify that

$$\begin{aligned} \frac{\partial \bar{\psi}}{\partial T} &\geqslant \frac{\sigma_2^2}{2}\frac{\partial^2 \bar{\psi}}{\partial r^2} - c_1(r - \bar{r})\frac{\partial \bar{\psi}}{\partial r} + [\gamma Q_M(r) - \beta_1]\bar{\psi}, \\ \bar{\psi}(r, 0) &\geqslant 1. \end{aligned}$$

Define $\xi(r, T) \equiv \psi(r, T) - \bar{\psi}(r)$. Then it satisfies

$$\begin{aligned} \frac{\partial \xi}{\partial T} &\leqslant \frac{\sigma_2^2}{2}\frac{\partial^2 \xi}{\partial r^2} - c_1(r - \bar{r})\frac{\partial \xi}{\partial r} + [\gamma Q_M(r) - \beta_1]\xi, \\ \xi(r, 0) &\leqslant 0. \end{aligned}$$

Since Q_M is bounded, there must exist a constant $B > 0$ such that $\gamma Q_M(r) - \beta_1 < B$. Define

$$\tilde{\xi}(r, T) \equiv e^{-BT}\xi(r, T), \quad \tilde{Q}_M(r) \equiv \gamma Q_M(r) - \beta_1 - B.$$

Then $\tilde{Q}_M(r) < 0$ and $\tilde{\xi}$ satisfies

$$\begin{aligned} \frac{\partial \tilde{\xi}}{\partial T} &\leqslant \frac{\sigma_2^2}{2}\frac{\partial^2 \tilde{\xi}}{\partial r^2} - c_1(r - \bar{r})\frac{\partial \tilde{\xi}}{\partial r} + \tilde{Q}_M(r)\tilde{\xi}, \\ \tilde{\xi}(r, 0) &\leqslant 0. \end{aligned}$$

Since $Q_M(r)$ is bounded, by definitions of $\psi, \bar{\psi}, \xi$ and $\tilde{\xi}$, we can get

$$\lim_{|r|\to\infty} \tilde{\xi}(r, T) = -\infty.$$

If $\tilde{\xi}(r, T)$ reaches its maximum on $\mathbf{R} \times [0, T_1]$ at a point (r_0, T_0), such that $\tilde{\xi}(r_0, T_0) > 0$, then we must have

$$\tilde{\xi}_r(r_0, T_0) = 0, \quad \tilde{\xi}_{rr}(r_0, T_0) \leqslant 0, \quad \tilde{\xi}_T(r_0, T_0) \geqslant 0.$$

This contradicts

$$\frac{\partial \tilde{\xi}}{\partial T} \leqslant \frac{\sigma_2^2}{2}\frac{\partial^2 \tilde{\xi}}{\partial r^2} - c_1(r-\bar{r})\frac{\partial \tilde{\xi}}{\partial r} + \tilde{Q}_M(r)\tilde{\xi}.$$

Therefore, we must have

$$\tilde{\xi}(r,T) \leqslant 0, \quad \forall r, T.$$

By definitions of $\tilde{\xi}$ and ξ, we can get

$$\psi(r,T) \leqslant \bar{\psi}(r), \quad \forall r, T.$$

Define $\Lambda(r) \equiv \bar{\psi}(r)$. Then $\Lambda(r)$ is a constant which does not depend on $M, \hat{M}$ or T. Thus, by the monotone convergence theorem, we can get (90). □

As a corollary, we have

Lemma 5.4 Suppose a is a constant which satisfies (84). If

$$\beta > -C_2(a), \tag{92}$$

then

$$\hat{W}(r) \leqslant A_1 e^{ar^2}, \tag{93}$$

where A_1 is a constant.

Proof. For any admissible control c_t, we can assume that $0 \leqslant c_t \leqslant \Lambda$, where Λ is a general constant. Therefore, by the definition of $\Phi(r,t)$, we have

$$\begin{aligned}\hat{W}(r) &= \sup_{u.,c.} \frac{1}{\gamma}\int_0^\infty e^{-\beta t}\hat{E}_r c_t^\gamma \exp\left\{\int_0^t \gamma[l(r_s,u_s)-c_s]ds\right\} dt \\ &\leqslant \frac{1}{\gamma}\Lambda^\gamma \int_0^\infty e^{-\beta t}\Phi(r,t)dt.\end{aligned}$$

Then (93) follows by virtue of Lemma 5.3. □

Lemma 5.5 If

$$\beta > \gamma b - \frac{\sigma_1^2}{2}\gamma(1-\gamma), \tag{94}$$

then $\hat{W}(r)$ is bounded away from 0.

Proof. Suppose

$$\delta_1 > \beta - \gamma b + \frac{\sigma_1^2}{2}\gamma(1-\gamma).$$

Take controls

$$\hat{u}_t \equiv 1, \quad \hat{c}_t \equiv \delta_1 - [\beta - \gamma b + \frac{\sigma_1^2}{2}\gamma(1-\gamma)].$$

Then by the definition of $\hat{W}(r)$, we can get

$$\hat{W}(r) \geqslant \hat{J}(r, \hat{u}., \hat{c}.).$$

Actually, we have

$$\begin{aligned}\hat{J}(r, \hat{u}., \hat{c}.) \qquad &= \hat{\Lambda}\int_0^\infty \hat{E}_r e^{\int_0^t -[\beta-\gamma b+\frac{\sigma_1^2}{2}\gamma(1-\gamma)+\hat{c}_s]ds}dt \\ &= \hat{\Lambda}\int_0^\infty e^{-\delta_1 t}dt = \frac{\hat{\Lambda}}{\delta} > 0,\end{aligned}$$

where $\hat{\Lambda} > 0$ is a constant. Thus, $\hat{W}(r)$ is bounded away from 0. □

Lemma 5.6 $\hat{W}(r)$ is convex and continuous in r.

Proof. By virtue of (73) – (74), it is easy to show that if $r_{1,t}$ and $r_{2,t}$ are solutions corresponding to initial conditions r_1 and r_2 respectively, then $\lambda r_{1,t} + (1-\lambda)r_{2,t}$ is the solution corresponding to the initial condition $\lambda r_1 + (1-\lambda)r_2$. Since $l(r, u)$ is linear with respect to r, noting that e^x is convex, we have

$$\begin{aligned} & \hat{J}(\lambda r_1 + (1-\lambda)r_2, u., c.) \\ = \ & \frac{1}{\gamma}\int_0^\infty e^{-\beta t}\hat{E}c_t^\gamma \exp\left\{\int_0^t \gamma[l(\lambda r_{1s} + (1-\lambda)r_{2s}, u_s) - c_s]ds\right\}dt \\ \leqslant \ & \frac{1}{\gamma}\int_0^\infty e^{-\beta t}\hat{E}c_t^\gamma \left[\lambda \exp\left\{\int_0^t \gamma(l(r_{1s}, u_s) - c_s)ds\right\}\right. \\ & \qquad \left. + (1-\lambda)\exp\left\{\int_0^t \gamma(l(r_{2s}, u_s) - c_s)ds\right\}\right]dt \\ = \ & \lambda\hat{J}(r_1, u., c.) + (1-\lambda)\hat{J}(r_2, u., c.) \\ \leqslant \ & \lambda\hat{W}(r_1) + (1-\lambda)\hat{W}(r_2).\end{aligned}$$

That is, $\hat{W}(r)$ is convex in r and then it is continuous in r. □

We know that the Girsanov transformation does not change the law of r_t, and r_t does not depend on the value of u_t. In addition, since $l(r, u)$ is quadratic with respect to u, by the definitions of $l(r, u)$ and $Q(r)$ (see(15), (25)), we can verify that

$$l(r, u) \leqslant Q(r), \qquad l(r, u^*) = Q(r),$$

where u^* is given by (76). Therefore, by the definition of $\hat{W}$ and $\hat{J}$, we can obtain that

$$\hat{W}(r) = \sup_{c.} \frac{1}{\gamma}\int_0^\infty e^{-\beta t}E_r c_t^\gamma \exp\left\{\int_0^t \gamma[Q(r_s) - c_s]ds\right\}dt. \tag{95}$$

Since our admissible control defined here is called feedback control in Borkar (Ref. 12), following the same method in the proof of Theorem 1.1 on page 56 of Ref. 12, we can get

Lemma 5.7 (Dynamic Programming Principle) Define τ_R the exit time of process r_t from the compact set $\{r : |r| \leqslant R\}$. Then the following hold:

(a) For any admissible control c_t, we have

$$\hat{W}(r) \geqslant E_r\left[\frac{1}{\gamma}\int_0^{\tau_R} e^{-\beta t}c_t^{\gamma}\exp\left\{\int_0^t \gamma[Q(r_s)-c_s]ds\right\}dt \right.$$
$$\left. + \exp\left\{\int_0^{\tau_R}[\gamma Q(r_t)-\gamma c_t-\beta]dt\right\}\hat{W}(r_{\tau_R})\right].$$

(b) $\forall \epsilon > 0$, there exists an admissible control c_t such that

$$\hat{W}(r)-\epsilon \leqslant E_r\left[\frac{1}{\gamma}\int_0^{\tau_R} e^{-\beta t}c_t^{\gamma}\exp\left\{\int_0^t \gamma[Q(r_s)-c_s]ds\right\}dt \right.$$
$$\left. + \exp\left\{\int_0^{\tau_R}[\gamma Q(r_t)-\gamma c_t-\beta]dt\right\}\hat{W}(r_{\tau_R})\right].$$

On the other hand, since $\hat{W}(r)$ is well defined for all $r \in \mathbf{R}$, we can defined a new value function $\tilde{W}(r)$ as

$$\tilde{W}(r) \equiv \sup_{c.} E_r\left[\frac{1}{\gamma}\int_0^{\tau_R} e^{-\beta t}c_t^{\gamma}\exp\left\{\int_0^t \gamma[Q(r_s)-c_s]ds\right\}dt \right.$$
$$\left. + \exp\left\{\int_0^{\tau_R}[\gamma Q(r_t)-\gamma c_t-\beta]dt\right\}\hat{W}(r_{\tau_R})\right], \tag{96}$$

where τ_R is the exit time of process r_t from the compact set $\{r : |r| \leqslant R\}$. By virtue of Lemma 5.7, we can get that

$$\tilde{W}(r) \equiv \hat{W}(r), \quad \forall r \in [-R, R].$$

Since $\hat{W}(r)$ is a continuous function on $\mathbf{R}$, we know that $\tilde{W}(r)$ is a continuous function on $(-R, R)$. Noting that c_t is bounded, by virtue of Lions (Ref. 13 Theorem I.1, page 1242), we get the following lemma

Lemma 5.8 $\tilde{W}(r)$ is a viscosity solution of (75) on $(-R, R)$.

By virtue of the fact that $\tilde{W}(r) \equiv \hat{W}(r)$ on $(-R, R)$ and R is arbitrary, we can show the following lemma:

Lemma 5.9 $\hat{W}(r)$ is a viscosity solution of (75) on the real line.

Next we will show that (75) has at most one viscosity solution. To do that, we define

$$W(r) \equiv \gamma \hat{W}(r). \tag{97}$$

Then $W(r)$ is a viscosity solution of

$$\frac{1}{2}\sigma_2^2 W_{rr} - c_1(r-\bar{r})W_r + [\gamma Q(r)-\beta]W + (1-\gamma)W^{\frac{\gamma}{\gamma-1}} = 0, \tag{98}$$

where $Q(r) = \frac{(b-r)^2}{2(1-\gamma)\sigma_1^2} + r$ is given by (25).

Now, according to Bailey, Shampine and Waltman (Ref. 14 Theorem 3.3, page 34), for any $r \in \mathbf{R}$, there exists an interval $[r_1, r_2]$ with $r \in (r_1, r_2)$ such that equation (98) with any given boundary condition has a unique classical solution on $[r_1, r_2]$. Since classical solutions are also viscosity solutions, if we have the uniqueness of the viscosity solution for the corresponding boundary problem on $[r_1, r_2]$, then $W(r)$ is also the unique classical solution of (98).

Suppose

$$Z(r) = \log W(r). \tag{99}$$

Then the equation of $Z(r)$ is

$$\frac{1}{2}\sigma_2^2(Z_{rr} + Z_r^2) - c_1(r - \bar{r})Z_r + \gamma Q(r) - \beta + (1-\gamma)e^{\frac{Z}{\gamma-1}} = 0. \tag{100}$$

Define

$$F(r, x, p, X) = -\frac{1}{2}\sigma_2^2(X + p^2) + c_1(r - \bar{r})p - \gamma Q(r) + \beta - (1-\gamma)e^{\frac{x}{\gamma-1}}, \tag{101}$$

then the equation of Z can be written as

$$F(r, Z, Z_r, Z_{rr}) = 0. \tag{102}$$

To show the uniqueness of W on a given finite interval $[R_1, R_2]$ with boundary condition $W(R_1) = W_1, W(R_2) = W_2$, it is sufficient to show the uniqueness of the solution of (102) with boundary condition

$$Z(R_1) = \log W_1, \quad Z(R_2) = \log W_2. \tag{103}$$

It is easy to check that F satisfies

$$F(r, x, p, X) \leqslant F(r, y, p, Y), \quad \forall x \leqslant y, Y \leqslant X.$$

In addition, for $y \geqslant x$, there is a $\xi \in [x, y]$ such that

$$\begin{aligned} F(r, y, p, X) - F(r, x, p, X) &= (\gamma-1)e^{\frac{y}{\gamma-1}} - (\gamma-1)e^{\frac{x}{\gamma-1}} \\ &= e^{\frac{\xi}{\gamma-1}}(y - x). \end{aligned}$$

Noting that $\gamma - 1 < 0$, if $x, y \leqslant \Lambda < \infty$, there exists $\alpha > 0$ such that

$$F(r, y, p, X) - F(r, x, p, X) \geqslant \alpha(y - x).$$

By Lemma 5.4 and Lemma 5.5, we can show that W is bounded away from 0 and it is bounded on any finite interval. Thus Z is bounded on $[R_1, R_2]$. Then we only need to consider the case when the solutions are bounded.

Finally, for a constant $\alpha_1 > 0$, if $|X|, |Y| < 3\alpha_1$, then

$$\begin{aligned} & F(r_1, x, \alpha_1(r_1 - r_2), Y) - F(r_2, x, \alpha_1(r_1 - r_2), X) \\ = \; & \frac{1}{2}\sigma_2^2(X - Y) + \gamma[Q(r_1) - Q(r_2)] + c_1\alpha_1(r_1 - r_2)^2 \\ \leqslant \; & \Lambda_1[(r_1 - r_2)^2 + (r_1 - r_2)], \end{aligned}$$

where $\Lambda_1 > 0$ is a constant. Now, according to Crandall, Ishii and Lions (Ref. 15 Theorem 3.3, page 18), (102) – (103) has only one viscosity solution. Thus, we have shown that

Lemma 5.10 Equation (98) with boundary condition

$$W(R_1) = W_1, \quad W(R_2) = W_2$$

has only one viscosity solution.

Given this lemma, now we know that $W(r)$ is in fact a classical solution of (98). Actually, we have proved that

Theorem 5.1 Let $0 < \gamma < \gamma_1$, where γ_1 is defined by (81). Defined $\hat{W}(r)$ as in (19) and suppose (94) holds. Then the following hold:
(a) $\hat{W}(r)$ is bounded away from 0, and it satisfies

$$\hat{W}(r) \leqslant A_1 e^{ar^2}, \tag{104}$$

where A_1 and a are positive constants and a satisfies (84).
(b) $\hat{W}(r)$ is a classical solution of (75).

Remark 5.1 Unfortunately, the techniques we use above do not apply to $\gamma < 0$ case, for that we can not get similar results as we obtained in Lemma 5.6.

References

1. FLEMING, W. H. and SONER, H. M., *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, NY, 1992.
2. FLEMING, W. H. and STEIN, J. L., *Stochastic Optimal Control in International Finance and Debt*, to appear in Journal of Banking and Finance.
3. FLEMING, W. H. and HERNANDEZ-HERNANDEZ, D., *An Optimal Consumption Model with Stochastic Volatility*, Finance and Stochastics, Vol. 7, pp 245-262, 2003.
4. FLEMING, W. H. and SHEU, S. J., *Optimal Long Term Growth Rate of Expected Utility of Wealth*, Annals of Applied Probability, Vol. 9, pp 871-903, 1999.
5. PANG, T., *Stochastic Control Theory and Its Applications to Financial Economics*, Ph.D thesis, Brown University, Providence, Rhode Island, 2002.
6. BIELECKI, T. R. and PLISKA, S. R., *Risk Sensitive Dynamic Asset Management*, Applied Mathematics and Optimization, Vol. 39, pp 337-360, 1999.
7. ZARIPHOPOULOU, T., *A Solution Approach to Valuation with Unhedgeable Risks*, Finance and Stochastics, Vol. 5, pp 61-82, 2001.
8. FLEMING, W. H. and PANG, T., *An Application of Stochastic Control Theory to Financial Economics*, to appear in SIAM Journal of Control and Optimization.
9. ÜSTÜNEL, A. S. and ZAKAI, M., *Transformation of Measure of Weiner Space*, Springer, New York, NY, 2000.

10. PAO, C. V., *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, NY, 1992.

11. FRIEDMAN, A., *Stochastic Differential Equations and Applications*, Vol. 1, Academic Press, New York, 1975.

12. BORKAR, V. S., *Optimal Control of Diffusion Processes*, Longman Scientific and Technical, Essex, UK, 1989.

13. LIONS, P.-L., *Optimal Control of Diffusion Processes and Hamiltion-Jacobi-Bellman Equations, Part 2: Viscosity Solutions and Uniqueness*, Communications in Partial Differential Equations, Vol. 8, pp 1229-1276, 1983.

14. BAILEY, P. B., SHAMPINE, L. F. and WALTMAN, P. E., *Nonlinear Two Point Boundary Value Problems*, Academic Press, New York, NY, 1968.

15. CRANDALL, M. G., ISHII, H., and LIONS, P. L., *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations*, Bulletin of the AMS, Vol. 27, pp 1-67, 1992.