Finite Difference Approximations for Stochastic Control Systems with Delay *

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Abstract

This paper considers the computation issues of the infinite dimensional HJB equation arising from the finite horizon optimal control problem of a general system of stochastic functional differential equations with a bounded memory treated in ([2]). The finite difference scheme, using the result in ([1]), is obtained to approximate the viscosity solution of the infinite dimensional HJB equation. The convergence of the scheme is proved using the Banach fixed point theorem. The computational algorithm is also provided based on the scheme obtained.

Keywords: Stochastic control, stochastic functional differential equations, viscosity solutions, finite difference approximation.

AMS 2000 subject classifications: primary 93C23, 34K28; secondary 34K50, 49L25

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1 Introduction

It has been recognized in recent years that the description of many real world problems can only be modelled by stochastic dynamical systems whose evolution depend on the past history of the state (see e.g. Kolmanovskii and Shaikhet [7]). Such models are often referred to as stochastic (retarded) functional differential equations (see Mohammed [12], [13] for an introduction of these models). Due to their applications in engineering, life science, finance, and other areas (see e.g. Mohammed [12]), its corresponding optimal control problems have recently received increasing research attention. The linear-quadratic regulatory problem involving stochastic delay equations was first studied in Kolmanovskii and Maizenberg [6], and optimal control problems for a class of nonlinear stochastic equations that involve a continuous delay of the following type

$$\begin{aligned} dX(s) &= \alpha(s, X(s), Y(s), u(s))ds \\ &\quad + \beta(s, X(s), Y(s), u(s))dW(s), \quad s \in [t, T], \end{aligned} \tag{1}$$

have been studied in recent literature (see e.g. Elsanousi [3], Elsanousi et al [4], and Larssen [10], Oksendal and Sulem [14]), in which $Y(s) = \int_{-r}^{0} e^{-\delta\theta} X(s+\theta)d\theta$.

In authors' recent research efforts (see Chang et al [2]), the finite horizon optimal control problem has been investigated for the general (retarded) stochastic functional differential equations of the following form:

$$\begin{aligned} dX(s) &= \alpha(s, X_s, u(s))ds \\ &\quad + \beta(s, X_s, u(s))dW(s), \quad s \in [t, T], \end{aligned} \tag{2}$$

where X_s is a $\mathbf{C}$-valued random variable for each $s \in [t, T]$ defined by $X_s(\theta) = X(s+\theta)$, $\theta \in [-r, 0]$, $r > 0$ denotes the duration of the bounded memory, and $\mathbf{C} \equiv C([-r, 0]; \mathbb{R}^n)$. In the above, the drift coefficient $\alpha(s, X_s, u(s))$ as well as the diffusion coefficient $\beta(s, X_s, u(s))$ at time $s \in [t, T]$ depend explicitly on the state variable over the time interval $[s-r, s]$ (and of course on the control variable $u(s)$). It is clear that (2) includes (1) as a special case. In the area of numerical computation, we mention here that the method of Markov chain approximation (see Kushner & Dupuis [9]) has been extended to solve an optimal control problem in which the state equation contains the general retardation in the state variable and delayed control in the drift term (see Kushner [8]).

The purpose of this paper is to investigate an implicit finite difference scheme for the optimal control problem stated in the next section. The finite difference scheme, deviating from that of Kusher [8], is an extension of that obtained by Barles and Songanidis [1]. This paper is organized as follows. The formulation of the finite horizon optimal control problem with the state equation described by

(2) as well as the infinite dimensional HJB equation are re-stated in Section 2. In Section 3, we investigate a finite difference scheme suitable for the the controlled stochastic functional differential equations. The convergence result of the approximation is also given in this section. The computational algorithm, based on the result obtained in Section 3, is summarized in Section 4.

2 The Infinite Dimensional HJB Equation

2.1 The Optimal Control Problem

Let $T > 0$ be the fixed terminal time, and let $t \in [0, T]$ be an initial time for the finite horizon stochastic optimal control problem considered in this paper. Let $r > 0$ be a fixed constant, and let $\mathbb{J} = [-r, 0]$ denote the duration of the bounded memory. Denote $C(\mathbb{J}; \mathbb{R}^n)$, the space of continuous functions $\phi : \mathbb{J} \to \mathbb{R}^n$, by $\mathbf{C}$. Note that $\mathbf{C}$ is a real separable Banach space under the sup-norm defined by

$$\|\phi\| = \sup_{t \in \mathbb{J}} |\phi(t)|, \quad \phi \in \mathbf{C},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. We denote by $(\,\cdot\,|\,\cdot\,)$ the inner product in $L^2(\mathbb{J}, \mathbb{R}^n)$, and $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$. Given ϕ and ψ in $\mathbf{C}$, we have defined as follows,

$$(\phi|\psi) = \int_{-r}^{0} \langle \phi(s), \psi(s) \rangle ds, \qquad \text{and } \; \|\phi\|_2 = (\phi|\phi)^{\frac{1}{2}}.$$

Note that the space $\mathbf{C}$ can be continuously embedded into $L^2(\mathbb{J}; \mathbb{R}^n)$.

Convention 2.1 *Throughout the end, we use the following conventional notation for functional differential equations (see Hale [5]): If* $\psi \in C([-r, \infty); \mathbb{R}^n)$ *and* $t \in \mathbb{R}_+$*, let* $\psi_t \in \mathbf{C}$ *be defined by* $\psi_t(\theta) = \psi(t + \theta)$*,* $\theta \in \mathbb{J}$*.*

Let $\{W(t), t \geq 0\}$ be a certain m-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}(t), t \geq 0\}$ is the P-augmentation natural filtration generated by the Brownian motion $\{W(t), t \geq 0\}$.

Consider the following system of controlled stochastic functional differential equations with a bounded memory:

$$dX(s) = f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad s \in [t, T], \tag{3}$$

and with the initial function $X_t = \psi$, where $\psi \in \mathbf{C}$ and $u(\cdot) = \{u(s), s \in [t,T]\}$ is a control process taking values in a compact set U (of an Euclidean space). The functions, $f : [0,T] \times \mathbf{C} \times U \to \mathbb{R}^n$ and $g : [0,T] \times \mathbf{C} \times U \to \mathbb{R}^{n\times m}$ are given deterministic functions.

Definition 2.2 *For each $t \in [0,T]$, a 5-tuples $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ is said to be an admissible control if it satisfies the following conditions:*

1. *$(\Omega, \mathcal{F}, P)$ is a complete probability space.*
2. *$W(\cdot) = \{W(s), s \in [0,T]\}$ is an m-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P)$ over $[t,T]$ with $W(t) = 0$ a.s., and $\mathcal{F}(t,s) = \sigma\{W(\tau), t \leq \tau \leq s\}$ augmented by the P-null sets in $\mathcal{F}$.*
3. *$u : [t,T] \times \Omega \to U$ is an $\{\mathcal{F}(t,s), s \in [t,T]\}$-adapted process on $(\Omega, \mathcal{F}, P)$ that is right-continuous at the initial time t.*
4. *Under the control process $u(\cdot) = \{u(s), s \in [t,T]\}$, equation (3) admits a unique strong solution $X^{t,\psi,u(\cdot)}(\cdot) = \{X(s; t, \psi, u(\cdot)), s \in [t,T]\}$ on $(\Omega, \mathcal{F}, P; \{\mathcal{F}(t,s), s \in [t,T]\})$ through each initial datum $(t,\psi) \in [0,T] \times \mathbf{C}$.*
5. *The control process $u(\cdot)$ is such that*

$$\mathbb{E}\left[\int_t^T |L(s, X_s(t,\psi,u(\cdot)), u(s))|ds + |\Psi(X_T(t,\psi,u(\cdot)))|\right] < \infty,$$

   where $L : [0,T] \times \mathbf{C} \times U \to \mathbb{R}$ and $\Psi : \mathbf{C} \to \mathbb{R}$ represent the running and terminal cost functions, respectively.

The collection of *admissible controls* $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ over the interval $[t,T]$ shall be denoted by $\mathcal{U}[t,T]$. We shall write $u(\cdot) \in \mathcal{U}[t,T]$ or $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot)) \in \mathcal{U}[t,T]$ interchangeably, whenever there is no danger of ambiguity.

Throughout the end, we assume that $f : [0,T] \times \mathbf{C} \times U \to \mathbb{R}^n$, and $g : [0,T] \times \mathbf{C} \times U \to \mathbb{R}^{n\times m}$ are continuous functions and satisfy the following linear growth and Lipschitz conditions. (See Mohammed [12, 13].)

Assumption 2.3 *There exists a constant $\Lambda > 0$ such that*

$$|f(t,\varphi,u) - f(t,\phi,u)| + |g(t,\varphi,u) - g(t,\phi,u)| \leq \Lambda\|\varphi - \phi\|,$$
$$\forall (t,\varphi,u), (t,\phi,u) \in [0,T] \times \mathbf{C} \times U.$$

Assumption 2.4 *There exists a constant $K > 0$ such that*

$$|f(t,\phi,u)| + |g(t,\phi,u)| \le K(1 + \|\phi\|), \ \forall (t,\phi,u) \in [0,T] \times \mathbf{C} \times U.$$

Given an admissible control $u(\cdot) \in \mathcal{U}[t,T]$, let $X^{t,\psi,u(\cdot)}(\cdot) = \{X(s;t,\psi,u(\cdot)), s \in [t,T]\}$ be the solution of (3) through the initial datum $(t,\psi) \in [0,T] \times \mathbf{C}$. We again consider the corresponding $\mathbf{C}$-valued process $\{X_s(t,\psi,u(\cdot)), s \in [t,T]\}$ defined by

$$X_s(\theta;t,\psi,u(\cdot)) = X(s+\theta;t,\psi,u(\cdot)), \quad \theta \in \mathbb{J}. \tag{4}$$

For notational simplicity, we often write $X(s) = X(s;t,\psi,u(\cdot))$ and $X_s = X_s(t,\psi,u(\cdot))$ for $s \in [t,T]$ whenever there is no danger of ambiguity.

It can be shown under Assumptions 2.3-2.4 that the $\mathbf{C}$-valued process $\{X_s(t,\psi,u(\cdot)), s \in [t,T]\}$ is a Markov process (see Mohammed [12], [13]). Let L and Ψ be two continuous real-valued functions on $[0,T] \times \mathbf{C} \times U$ and $[0,T] \times \mathbf{C}$, respectively. Moreover, we assume that they both have at most polynomial growth in $L^2(\mathbb{J};\mathbb{R})$. In other words, there exist constants Λ, k such that

$$|L(t,\phi,u)| \le \Lambda(1 + \|\phi\|_2)^k \quad \text{and} \quad |\Psi(t,\phi)| \le \Lambda(1 + \|\phi\|_2)^k,$$

for all $(t,\phi,u) \in [0,T] \times \mathbf{C} \times U$, for some positive integer k. Given any initial data $(t,\psi) \in [0,T] \times \mathbf{C}$ and any admissible control $u(\cdot) \in \mathcal{U}[t,T]$, we define the objective function

$$\begin{aligned} J(t,\psi;u(\cdot)) \equiv \mathbb{E}\bigg[\int_t^T & e^{-\rho(s-t)} L(s, X_s(t,\psi,u(\cdot)), u(s))ds \\ & + e^{-\rho(T-t)} \Psi(X_T(t,\psi,u(\cdot))) \bigg], \end{aligned} \tag{5}$$

where $\rho > 0$ denotes a discount factor. For each initial datum $(t,\psi) \in [0,T] \times \mathbf{C}$, the optimal control problem is to find $u(\cdot) \in \mathcal{U}[t,T]$ so as to maximize the objective function J. In this case, the value function $V : [0,T] \times \mathbf{C} \to \mathbb{R}$ is defined to be

$$V(t,\psi) = \sup_{u(\cdot) \in \mathcal{U}[t,T]} J(t,\psi;u(\cdot)). \tag{6}$$

2.2 The Equation

Let $\mathbf{C}^*$ and $\mathbf{C}^\dagger$ be the space of bounded linear functionals $\Phi : \mathbf{C} \to \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : \mathbf{C} \times \mathbf{C} \to \mathbb{R}$, of the space $\mathbf{C}$, respectively. They are equipped with the operator norms which will be, respectively, denoted by $\|\cdot\|^*$ and $\|\cdot\|^\dagger$.

Let $\mathbf{B} = \{v\mathbf{1}_{\{0\}}, v \in \mathbb{R}^n\}$, where $\mathbf{1}_{\{0\}} : [-r, 0] \to \mathbb{R}$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r, 0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$\mathbf{C} \oplus \mathbf{B} = \{\phi + v\mathbf{1}_{\{0\}} \mid \phi \in \mathbf{C},\ v \in \mathbb{R}^n\}$$

and equip it with the norm $\|\cdot\|$ defined by

$$\|\phi + v\mathbf{1}_{\{0\}}\| = \sup_{\theta \in [-r,0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbf{C},\ v \in \mathbb{R}^n.$$

Note that for each sufficiently smooth function $\Phi : \mathbf{C} \to \mathbb{R}$, its first order Fréchet derivative (with respect to $\phi \in \mathbf{C}$), $D\Phi(\varphi) \in \mathbf{C}^*$, has a unique and continuous linear extension $\overline{D\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi) \in \mathbf{C}^\dagger$, has a unique and continuous linear extension $\overline{D^2\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^\dagger$. In above, $(\mathbf{C} \oplus \mathbf{B})^*$ and $(\mathbf{C} \oplus \mathbf{B})^\dagger$ are spaces of bounded linear and bilinear functionals of $\mathbf{C} \oplus \mathbf{B}$, respectively. (See Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [12] for details.)

Throughout the end, let $C^{1,2}_{lip}([0,T] \times \mathbf{C})$ be the space of functions $\Phi : [0,T] \times \mathbf{C} \to \Re$ such that $\frac{\partial \Phi}{\partial t} : [0,T] \times \mathbf{C} \to \Re$ and $D^2\Phi : [0,T] \times \mathbf{C} \to \mathbf{C}^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t,\phi) - D^2\Phi(t,\varphi)\|^\dagger \le K\|\phi - \varphi\| \quad \forall t \in [0,T],\ \phi, \varphi \in \mathbf{C}.$$

For a Borel measurable function $\Phi : \mathbf{C} \to \mathbb{R}$, we also define

$$\mathcal{S}(\Phi)(\phi) = \lim_{h \to 0+} \frac{1}{h}\left[\Phi(\tilde{\phi}_h) - \Phi(\phi)\right]$$

for all $\phi \in \mathbf{C}$, where $\tilde{\phi} : [-r, T] \to \mathbb{R}^n$ is an extension of ϕ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0) \\ \phi(0) & \text{if } t \ge 0, \end{cases}$$

and again $\tilde{\phi}_t \in \mathbf{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-r, 0].$$

Let $\mathcal{D}(\mathcal{S})$, the domain of the operator $\mathcal{S}$, be the set of functions $\Phi : \mathbf{C} \to \mathbb{R}$ such that $\mathcal{S}(\Phi)(\phi)$ exists for each $\phi \in \mathbf{C}$.

Let $v \in U$. We define:

$$\mathcal{A}^v V(t,\psi) \equiv \mathcal{S}(V)(t,\psi) + \overline{DV(t,\psi)}(f(t,\psi,v)\mathbf{1}_{\{0\}}) + \frac{1}{2}\sum_{i=1}^{m} \overline{D^2V(t,\psi)}(g(t,\psi,v)\mathbf{e}_i\mathbf{1}_{\{0\}}, g(t,\psi,v)\mathbf{e}_i\mathbf{1}_{\{0\}}).$$

In this paper, we assume that for every $v \in U$, the domain of the generator $\mathcal{A}^v$ is large enough to contain $C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$.

We have the following result:

Theorem 2.5 *The value function V defined by (6) is the unique viscosity solution of following HJB equation:*

$$\rho V(t,\psi) - \frac{\partial V}{\partial t}(t,\psi) - \max_{v\in U}[\mathcal{A}^v V(t,\psi) + L(t,\psi,v)] = 0 \tag{7}$$

on $[0,T] \times \mathbf{C}$, and $V(T,\psi) = \Psi(\psi)$, $\forall \psi \in \mathbf{C}$.

Proof. For the proof of this result one can refer to Chang *et al.* [2]. □

In the next section we propose a numerical scheme to approximate the solution of (7).

3 A Finite Difference Scheme

In this section, we consider an explicit finite difference scheme and show that it converges to the unique viscosity solution of equation (7). We will use a method introduced by Barles and Souganidis [1]. Given a positive integer M, we consider the following truncated optimal control problem with value function $V_M : [0,T] \times \mathbf{C} \to \mathbb{R}$

$$V_M(t,\psi) = \sup_{u(\cdot)\in\mathcal{U}[t,T]} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}(L(s,X_s,u(s)) \wedge M)ds + e^{-\rho(T-t)}(\Psi(X_T) \wedge M)\right], \tag{8}$$

where $a \wedge b$ is defined by $a \wedge b = \min\{a,b\}$ for all $a,b \in \mathbb{R}$.

The corresponding HJB equation is given by

$$\rho V_M(t,\psi) - \frac{\partial V_M}{\partial t}(t,\psi) - \max_{u\in U}[\mathcal{A}^u V_M(t,\psi) + L(t,\psi,u) \wedge M] = 0 \tag{9}$$

on $[0,T]\times \mathbf{C}$, and $V(T,\psi)=\Psi(\psi)\wedge M,\ \forall\psi\in\mathbf{C}$. The corresponding truncated Hamiltonian is

$$
\begin{aligned}
&\mathcal{H}_M(t,\psi,V_M(t,\psi),\frac{\partial}{\partial t}V_M(t,\psi),\overline{DV_M(t,\psi)},\overline{D^2V_M(t,\psi)})\\
=\ & \mathcal{S}(V_M)(t,\psi)+\frac{\partial V_M}{\partial t}(t,\psi)\\
&+\max_{u\in U}\Bigg[\overline{DV_M(t,\psi)}(f(t,\psi,u)\mathbf{1}_{\{0\}})+L(t,\psi,u)\wedge M\\
&\qquad +\frac{1}{2}\sum_{i=1}^{m}\overline{D^2V_M(t,\psi)}(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}},g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})\Bigg]. \qquad (10)
\end{aligned}
$$

Similarly as in [2] it can be shown that the value function V_M is the unique viscosity solution of the equation

$$
\begin{aligned}
\rho V_M(t,\psi)-\mathcal{H}_M(t,\psi,V_M(t,\psi),\frac{\partial}{\partial t}V_M(t,\psi),\overline{DV_M(t,\psi)},\overline{D^2V_M(t,\psi)})=0,\\
\text{on } [0,T]\times\mathbf{C},\ \text{ with } V(T,\psi)=\Psi(\psi)\wedge M,\ \forall\psi\in\mathbf{C}. \qquad (11)
\end{aligned}
$$

Moreover, it is easy to see that $V_M\to V$ as $M\to\infty$. In view of these, we need only find the numerical solution for V_M. Let ε with $0<\varepsilon<1$ be the stepsize for variable ψ and η with $0<\eta<1$ be the stepsize for t. We consider the finite difference operators Δ_ε, Δ_η and Δ_η^2 defined by

$$
\begin{aligned}
\Delta_\eta W(t,\psi) &= \frac{W(t+\eta,\psi)-W(t,\psi)}{\eta},\\
\Delta_\varepsilon W(t,\psi)(h+v\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(h+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon},\\
\Delta_\varepsilon^2 W(t,\psi)(h+v\mathbf{1}_{\{0\}},k+w\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(h+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2}\\
&+\frac{W(t,\psi-\varepsilon(k+w\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2}.
\end{aligned}
$$

where $h,k\in\mathbf{C}$ and $v,w\in\mathbb{R}^n$. Recall that,

$$\mathcal{S}(\Phi)(\phi)=\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].$$

Therefore we define,

$$\mathcal{S}_\varepsilon(\Phi)(\phi)=\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].$$

It is clear that $\mathcal{S}_\varepsilon(\Phi)$ is an approximation of $\mathcal{S}(\Phi)$.

We have the following lemma:

Lemma 3.1 *For any $W : [0,T] \times \mathbf{C} \to \mathbb{R}$, $W \in \mathcal{C}^{1,2}([0,T] \times \mathbf{C})$ such that W can be smoothly extended on $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$, we have*

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon W(t,\psi)(h + v\mathbf{1}_{\{0\}}) = \overline{DW(t,\psi)}(h + v\mathbf{1}_{\{0\}}), \tag{12}$$

and

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon^2 W(t,\psi)(h + v\mathbf{1}_{\{0\}}) = \overline{D^2W(t,\psi)}(h + v\mathbf{1}_{\{0\}}, k + w\mathbf{1}_{\{0\}}). \tag{13}$$

Proof. Note that the function W can be extended from $[0,T] \times \mathbf{C}$ to $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$. Let us denote by $\widetilde{W}$ the smooth extension of W on $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$. It is clear that $\lim_{\varepsilon \to 0} \Delta_\varepsilon W(t,\psi) = d_G\widetilde{W}(t,\psi)$ where $d_G\widetilde{W}$ denote the Gâteau derivative of $\widetilde{W}$ with respect to its second variable. And since $\widetilde{W}$ is smooth then the Gâteau derivative and the Fréchet derivative of $\widetilde{W}$ coincide and are continuous extension of the DW, the Fréchet derivative of W. The uniqueness of the linear continuous extension then we have result

$$\begin{aligned} \lim_{\varepsilon \to 0} \Delta_\varepsilon W(t,\psi)(h + v\mathbf{1}_{\{0\}}) &= \lim_{\varepsilon \to 0} \Delta_\varepsilon \widetilde{W}(t,\psi)(h + v\mathbf{1}_{\{0\}}) \\ &= \overline{DW(t,\psi)}(h + v\mathbf{1}_{\{0\}}). \end{aligned} \tag{14}$$

Similarly argument for can used for (13). □

Let $\varepsilon, \eta > 0$. The corresponding discrete version of equation (11) is given by

$$\begin{aligned} &\rho V_M(t,\psi) \\ = \; &\frac{1}{\varepsilon}\Big[V_M(t,\tilde{\psi}_\varepsilon) - V_M(t,\psi)\Big] + \frac{V_M(t+\eta,\psi) - V_M(t,\psi)}{\eta} \\ &+ \max_{u \in U}\Bigg[\frac{V_M(t,\psi + \varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}})) - V_M(t,\psi)}{\varepsilon} \\ &\qquad + \frac{1}{2}\sum_{i=1}^m \Bigg(\frac{V_M(t,\psi + \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) - V_M(t,\psi)}{\varepsilon^2} \\ &\qquad\qquad + \frac{V_M(t,\psi - \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) - V_M(t,\psi)}{\varepsilon^2}\Bigg) \\ &\qquad + L(t,\psi,u) \wedge M\Bigg]. \end{aligned} \tag{15}$$

Rearranging terms, we obtain

$$\max_{u \in U}\Bigg[\frac{1}{\varepsilon}V_M(t,\tilde{\psi}_\varepsilon) + \frac{V_M(t+\eta,\psi)}{\eta} + \frac{V_M(t,\psi + \varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon}$$

$$+\frac{1}{2}\sum_{i=1}^{m}\frac{V_M(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+V_M(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}$$
$$+L(t,\psi,u)\wedge M-(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho)V_M(t,\psi)\Bigg]=0. \tag{16}$$

Since the term $(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho)$ is always positive, (16) is equivalent to

$$\max_{u\in U}\Bigg[\frac{1}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho}\Bigg(\frac{1}{\varepsilon}V_M(t,\tilde{\psi}_\varepsilon)+\frac{V_M(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon}$$
$$+\frac{1}{2}\sum_{i=1}^{m}\frac{V_M(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+V_M(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}$$
$$+\frac{V_M(t+\eta,\psi)}{\eta}+\min(L(t,\psi,u),M)\Bigg)-V_M(t,\psi)\Bigg]=0. \tag{17}$$

Let $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ denote the space of bounded continuous functions W from $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ to $\mathbb{R}$. Define a mapping $S_M:(0,1)^2\times[0,T]\times\mathbf{C}\times\mathbb{R}^n\times C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\to\mathbb{R}$ as the following

$$S_M(\varepsilon,\eta,t,\psi,x,W)=\varepsilon\max_{u\in U}\Bigg[\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)+\frac{W(t+\eta,\psi)}{\eta}$$
$$+\frac{W(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon}+L(t,\psi,u)\wedge M$$
$$+\frac{1}{2}\sum_{i=1}^{m}\frac{W(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+W(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\Bigg]$$
$$-\varepsilon\left(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho\right)x. \tag{18}$$

Then, (15) is equivalent to $S_M(\varepsilon,\eta,t,\psi,V_M(t,\psi),V_M)=0$. Moreover, note that the coefficient of x in S_M is negative. This implies that S_M is monotone, i.e., for all $x_1,x_2\in\mathbb{R}^n$, $\varepsilon,\eta\in(0,1)$, $t\in[0,T]$, $\psi\in\mathbf{C}$, and $W\in C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$

$$S_M(\varepsilon,\eta,t,\psi,x_1,W)\le S_M(\varepsilon,\eta,t,\psi,x_2,W)\text{ whenever }x_1\ge x_2,$$

where $x_1\ge x_2$ denotes componentwise inequality.

Definition 3.2 The scheme S_M is said to be consistent if, for every $t\in[0,T]$, $\psi\in\mathbf{C}\oplus\mathbf{B}$, and for every test function $W(\cdot,\cdot)$ defined on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ such that $W \in C_b^{1,2}([0,T] \times (\mathbf{C} \oplus \mathbf{B})) \cap \mathcal{D}(\mathcal{S})$,

$$\begin{aligned}
&\rho W(t,\psi) - \mathcal{H}_M(t,\psi,W(t,\psi),\frac{\partial}{\partial t}W(t,\psi),\overline{DW(t,\psi)},\overline{D^2W(t,\psi)}) \\
&= \lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,\xi\to 0} \frac{S_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon}.
\end{aligned}$$

Lemma 3.3 *The scheme S_M is consistent.*

Proof. Let $W \in C_b^{1,2}([0,T] \times (\mathbf{C} \oplus \mathbf{B})) \cap \mathcal{D}(\mathcal{S})$. We write

$$\begin{aligned}
\frac{S_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon} &= \max_{u\in U}\Bigg[\Bigg(\frac{1}{\varepsilon}(W(\tau,\tilde{\psi}_\varepsilon)+\xi) \\
&+ \frac{W(\tau,\psi+\varepsilon(f(\tau,\psi,u)\mathbf{1}_{\{0\}}))+\xi}{\varepsilon} + L(\tau,\psi,u)\wedge M + \frac{W(t+\eta,\psi)+\xi}{\eta} \\
&+ \frac{1}{2}\sum_{i=1}^m \frac{W(\tau,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+2\xi+W(\tau,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\Bigg)\Bigg] \\
&- (\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho)(W(\tau,\phi)+\xi).
\end{aligned}$$

Sending $\xi \to 0$, $\tau \to t$, $\phi \to \psi$, $\epsilon,\eta \to 0$, we have

$$\begin{aligned}
&\rho W(t,\psi) - \mathcal{H}_M(t,\psi,W(t,\psi),\frac{\partial}{\partial t}W(t,\psi),\overline{DW(t,\psi)},\overline{D^2W(t,\psi)}) \\
&= \lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,\xi\to 0} \frac{S_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon}.
\end{aligned}$$

This completes the proof. □

Using (17), we see that the equation $S_M(\varepsilon,\eta,t,\psi,W(t,\psi),W)=0$ is equivalent to the equation

$$\begin{aligned}
W(t,\psi) &= \max_{u\in U}\Bigg[\frac{1}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho}\Bigg(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon) + \frac{W(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon} \\
&+ \frac{1}{2}\sum_{i=1}^m \frac{W(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+W(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
&+ \frac{W(t+\eta,\psi)}{\eta} + L(t,\psi,u)\wedge M\Bigg)\Bigg]. \qquad (19)
\end{aligned}$$

We define an operator $\mathcal{T}_{\varepsilon,\eta}$ on $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ as follows,

$$\begin{aligned}
&\mathcal{T}_{\varepsilon,\eta}W(t,\psi) \\
\equiv\ & \max_{u\in U}\Bigg[\frac{1}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho}\Bigg(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)+\frac{W(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon} \\
&\quad +\frac{1}{2}\sum_{i=1}^{m}\frac{W(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+W(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
&\quad +\frac{W(t+\eta,\psi)}{\eta}+L(t,\psi,u)\wedge M\Bigg)\Bigg]. \qquad (20)
\end{aligned}$$

Lemma 3.4 *For each ε and η, $\mathcal{T}_{\varepsilon,\eta}$ is a contraction map.*

Proof. To prove that $\mathcal{T}_{\epsilon,\eta}$ is a contraction, we need to show that there exists $0<\beta<1$ such that

$$\|\mathcal{T}_{\varepsilon,\eta}W_1-\mathcal{T}_{\varepsilon,\eta}W_2\|\le\beta\|W_1-W_2\| \ \text{ for all } \ W_1,W_2\in C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B})),$$

where $\|\cdot\|$ is the sup norm. Let us define $c_{\varepsilon,\eta}$ by

$$c_{\varepsilon,\eta}=\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho.$$

Note that

$$\begin{aligned}
&|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi)-\mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)| \\
&\le\max_{u\in U}\Bigg[\frac{1}{c_{\varepsilon,\eta}}\Bigg|\Bigg(\frac{1}{\varepsilon}W_1(t,\tilde{\psi}_\varepsilon)+\frac{W_1(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon}+\frac{W_1(t+\eta,\psi)}{\eta} \\
&+\frac{1}{2}\sum_{i=1}^{m}\frac{W_1(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+W_1(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\Bigg) \\
&-\Bigg(\frac{1}{\varepsilon}W_2(t,\tilde{\psi}_\varepsilon)+\frac{W_2(t,\psi+\varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon}+\frac{W_2(t+\eta,\psi)}{\eta} \\
&+\frac{1}{2}\sum_{i=1}^{m}\frac{W_2(t,\psi+\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))+W_2(t,\psi-\varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2}\Bigg)\Bigg|\Bigg].
\end{aligned}$$

This implies that for all t,ψ,

$$\Big|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi)-\mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)\Big|\le\left[\frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{c_{\varepsilon,\eta}}\right]\|W_1-W_2\|.$$

In addition, note that

$$\frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{c_{\varepsilon,\eta}}=\frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{m}{\varepsilon^2}+\rho}<1.$$

Take

$$\beta_{\varepsilon,\eta} = \frac{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2}}{c_{\varepsilon,\eta}}.$$

Therefore,

$$\|\mathcal{T}_{\varepsilon,\eta}W_1 - \mathcal{T}_{\varepsilon,\eta}W_2\| \leq \beta_{\varepsilon,\eta}\|W_1 - W_2\|.$$

□

Definition 3.5 The scheme S_M is said to be **stable** if for every $\varepsilon, \eta \in (0,1)$, there exists a bounded solution $W_{\varepsilon,\eta} \in C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B}))$ to the equation

$$S_M(\varepsilon, \eta, t, \psi, W(t,\psi), W) = 0, \tag{21}$$

with the bound independent of ε, and η.

Remark 3.6 By the Banach fixed point theorem, the strict contraction $\mathcal{T}_{\varepsilon,\eta}$ has a unique fixed point that we denote by $W^M_{\varepsilon,\eta}$. Given any function $W_0 \in \mathcal{C}([0,T] \times (\mathbf{C} \oplus \mathbf{B}))_b$, we construct a sequence as follows, $W_{n+1} = \mathcal{T}_{\varepsilon,\eta}W_n$ for $n \geq 0$. It is clear that

$$\lim_{n\to\infty} W_n = W^M_{\varepsilon,\eta}.$$

Moreover, note that

$$\begin{aligned}
& W_{n+1}(t,\psi) \\
= \; & \max_{u \in U}\left[\frac{1}{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho}\left(\frac{1}{\varepsilon}W_n(t,\tilde{\psi}_\varepsilon) + \frac{W_n(t, \psi + \varepsilon(f(t,\psi,u)\mathbf{1}_{\{0\}}))}{\varepsilon}\right.\right. \\
& + \frac{1}{2}\sum_{i=1}^{m}\frac{W_n(t,\psi + \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}})) + W_n(t,\psi - \varepsilon(g(t,\psi,u)\mathbf{e}_i\mathbf{1}_{\{0\}}))}{\varepsilon^2} \\
& \left.\left. + \frac{W_n(t+\eta,\psi)}{\eta} + L(t,\psi,u) \wedge M\right)\right] \\
\leq \; & \beta_{\varepsilon,\eta}\|W_n\| + \frac{1}{c_{\varepsilon,\eta}}M.
\end{aligned} \tag{22}$$

In addition, we have

$$\beta_{\varepsilon,\eta} = \frac{c_{\varepsilon,\eta} - \rho}{c_{\varepsilon,\eta}} < 1,$$

This implies that

$$\|W_{n+1}\| \leq \frac{c_{\varepsilon,\eta} - \rho}{c_{\varepsilon,\eta}}\|W_n\| + \frac{1}{c_{\varepsilon,\eta}}M. \tag{23}$$

From (23), we deduce that

$$\|W_{n+1}\| \leq \left(\frac{c_{\varepsilon,\eta}-\rho}{c_{\varepsilon,\eta}}\right)^{n+1} \|W_0\| + \frac{M}{c_{\varepsilon,\eta}} \sum_{i=0}^{n} \left(\frac{c_{\varepsilon,\eta}-\rho}{c_{\varepsilon,\eta}}\right)^{i}.$$

Taking the limit as $n \to \infty$, we obtain

$$\|W^M_{\varepsilon,\eta}\| \leq \frac{M}{c_{\varepsilon,\eta}} \cdot \frac{1}{1-\frac{c_{\varepsilon,\eta}-\rho}{c_{\varepsilon,\eta}}} = \frac{M}{\rho}.$$

This implies the stability of the scheme S_M.

Theorem 3.7 *Let $W^M_{\varepsilon,\eta}$ denote the solution to* (21)*. Then, as $(\varepsilon,\eta) \to 0$, the sequence $W^M_{\varepsilon,\eta}$ converges uniformly on $[0,T] \times \mathbf{C}$ to the unique viscosity solution V_M of* (11)*.*

Proof. Define

$$\begin{aligned} W^*_M(t,\psi) &= \limsup_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W^M_{\varepsilon,\eta}(\tau,\phi) \\ W_{*M}(t,\psi) &= \liminf_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W^M_{\varepsilon,\eta}(\tau,\phi). \end{aligned} \tag{24}$$

We claim that W^*_M and W_{*M} are sub- and supersolutions of (11), respectively. To prove this claim, we only consider the case for W^*_M. The argument for that of W_{*M} is similar. We want to show:

$$\rho\Gamma(t,\psi) - \mathcal{H}_M(t,\psi,\Gamma(t,\psi), \frac{\partial}{\partial t}\Gamma(t,\psi), \overline{D\Gamma(t,\psi)}, \overline{D^2\Gamma(t,\psi)}) \leq 0,$$

for any test function $\Gamma \in C^{1,2}_{lip}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$ such that (t,ψ) is a strictly local maximum of $W^*_M(\tau,\phi)-\Gamma(\tau,\phi)$. Without loss of generality, we may assume that $W^*_M \leq \Gamma$ and $W^*_M(\tau,\phi) = \Gamma(\tau,\phi)$ and because of the stability of our scheme we can also assume that $\Gamma \geq 2\sup_{\varepsilon,\eta}\|W^M_{\varepsilon,\eta}\|$ outside of the ball $B((t,\psi),l)$ where $l>0$ is such that

$$W^*_M(\tau,\phi) - \Gamma(\tau,\phi) \leq 0 = W^*_M(t,\psi) - \Phi(t,\psi) \text{ for } (\tau,\phi) \in B((t,\psi),l).$$

This implies that there exist sequences $\varepsilon_n > 0$, $\eta_n > 0$, and $(\tau_n,\phi_n) \in [0,T] \times (\mathbf{C}\oplus\mathbf{B})$ such that as $n\to\infty$ we have

$$\begin{aligned} &\varepsilon_n \to 0,\ \ \eta_n \to 0,\ \ \tau_n \to t,\ \ \phi_n \to \psi,\ \ W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n) \to W^*_M(t,\psi), \\ &\text{and } (\tau_n,\phi_n) \text{ is a global maximum } W^M_{\varepsilon_n,\eta_n} - \Gamma. \end{aligned} \tag{25}$$

Denote $\alpha_n = W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n) - \Gamma(\tau_n,\phi_n)$. Obviously $\alpha_n \to 0$ and

$$W^M_{\varepsilon_n,\eta_n}(\tau,\phi) \leq \Gamma(\tau,\phi) + \alpha_n \quad \text{for all} \quad (\tau,\phi) \in [0,T] \times (\mathbf{C} \oplus \mathbf{B}). \tag{26}$$

We know that

$$S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n}) = 0.$$

The monotonicity of S_M and (26) implies

$$\begin{aligned} & S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, \Gamma(\tau_n,\phi_n) + \alpha_n, \Gamma(\tau_n,\phi_n) + \alpha_n) \\ \leq \; & S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n}) \\ = \; & 0. \end{aligned} \tag{27}$$

Therefore,

$$\lim_{n\to\infty} \frac{S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n})}{\varepsilon_n} \leq 0,$$

so

$$\begin{aligned} & \rho W^*_M(t,\psi) - \mathcal{H}_M(t,\psi,W^*_M(t,\psi), \mathrm{D}_t\Gamma(t,\psi), \overline{D\Gamma(t,\psi)}, \overline{D^2\Gamma(t,\psi)}) \\ & = \lim_{n\to\infty} \frac{S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n})}{\varepsilon_n} \leq 0. \end{aligned} \tag{28}$$

This proves that W^*_M is a viscosity subsolution of (11) and, similarly we can prove that W_{*M} is a viscosity supersolution. By virtue of Theorem 5.1 in [2], we can get that

$$W_{*M}(t,\psi) \geq W^*_M(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}. \tag{29}$$

On the other hand, by the definition of W_{*M}, W^*_M, it is easy to see that

$$W_{*M}(t,\psi) \leq W^*_M(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Combined with (29), the above implies

$$W_{*M}(t,\psi) = W^*_M(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Since W_{*M} is a viscosity supersolution and W^*_M is a viscosity subsolution, they are also viscosity solutions of (11). Now, using the uniqueness of the viscosity solution (11), we see that $V_M = W^*_M = W_{*M}$. Therefore, we conclude that the sequence $(W^M_{\varepsilon,\eta})_{\varepsilon,\eta}$ converges locally uniformly to V_M as desired. □

4 The Computational Algorithm

Based on the results obtained in the last section, we can construct the computational algorithm to obtain a numerical solution. For example, one algorithm can be like the following:

Step 0. Choose any function $W^{(0)} \in C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$;

Step 1. Pick the starting values for $\epsilon(1), \eta(1)$. For example, we can choose $\epsilon(1) = 10^{-2}, \eta(1) = 10^{-3}$;

Step 2. For the given $\epsilon, \eta > 0$, compute the function

$$W^{(1)}_{\epsilon(1),\eta(1)} \in C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$$

by the following formula

$$W^{(1)}_{\epsilon(1),\eta(1)} = \mathcal{T}_{\epsilon(1),\eta(1)} W^{(0)},$$

where $\mathcal{T}_{\epsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (20);

Step 3. Repeat Step 2 for $i = 2, 3, \cdots$ using

$$W^{(i)}_{\epsilon(1),\eta(1)} = \mathcal{T}_{\epsilon(1),\eta(1)} W^{(i-1)}_{\epsilon(1),\eta(1)},$$

where $\mathcal{T}_{\epsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (20). Stop the iteration when

$$\|W^{i+1}_{\epsilon(1),\eta(1)}(t,\psi) - W^{i}_{\epsilon(1),\eta(1)}(t,\psi)\| \leq \delta_1,$$

where δ_1 is a preselected number which is small enough to achieve the accuracy we want. Denote the final solution by $W_{\epsilon(1),\eta(1)}(t,\psi)$.

Step 4. Choose two sequences of $\epsilon(k)$ and $\eta(k)$, such that

$$\lim_{k\to\infty} \epsilon(k) = \lim_{k\to\infty} \eta(k) = 0.$$

For example, we may choose $\epsilon(k) = \eta(k) = 10^{-(2+k)}$. Now repeat Step 2 and Step 3 for each $\epsilon(k), \eta(k)$ until

$$\|W_{\epsilon(k+1),\eta(k+1)}(t,\psi) - W_{\epsilon(k),\eta(k)}(t,\psi)\| \leq \delta_2,$$

where δ_2 is chosen to obtain the expected accuracy.

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