An Application of Functional Ito’s Formula to Stochastic Portfolio Optimization with Bounded Memory

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Abstract

We consider a stochastic portfolio optimization model in which the returns of risky asset depend on its past performance. The price of the risky asset is described by a stochastic delay differential equation. The investor’s goal is to maximize the expected discounted utility by choosing optimal investment and consumption as controls. We use the functional Ito’s formula to derive the associated Hamilton-Jacobi-Bellman equation. For logarithmic and exponential utility functions, we can obtain explicit solutions in a finite dimensional space.

1 Introduction.

In a classical portfolio optimization of Merton’s type, an investor allocates her wealth between a risky asset and a riskless asset and choose the consumption rate to maximize the total expected utility. The price of the risky asset is usually described by a Markovian stochastic process, such as geometric Brownian motion. In such cases, the investor will make the decision only based on the current information, and the historic information does not make any difference. The related problems have been studied by many researchers. Some related literatures include [8], [9], [10], [11], [19] and the references therein.

However, in the real world, the stock price process may not follow a Markovian process. For example, investors tend to look at the historical performance of a stock before they invest on the stock. So a stock with strong historical performance may attract more investors and that will increase the demand of the stock. Therefore, the price of a stock with strong historical performance tends to increase. This suggests that the price of a stock should be modeled by a stochastic process with memory or delay.

In this paper, we consider a stochastic portfolio management problem taking into account the history of the portfolio performance. Investor’s portfolio consists of a risky and a riskless asset. A stochastic delayed equation is used to describe the value change of the investor’s portfolio. The value of the portfolio follows a stochastic process \( X(t) \) that is given by (2.7) and it depends on delay variables given by (2.1)-(2.2). The performance of the portfolio in \( [s-h, s] \) gives the initial condition for \( X(t) \) and is characterized by (2.4).

In general the solution of a stochastic control problem with delay may depend on the initial condition \( \varphi \), which is in the space of all continuous function on \( C[-h, 0] \)-an infinite dimensional space. However, if the system only depends on the delay through process \( Y(t) \) and \( Z(t) \) defined by (2.1)-(2.2), it is possible to obtain a solution in a finite dimensional space, such as in Chang, Pang and Yang [2], in which HARA utility function was considered. In this paper, we will consider this model for logarithmic utility function and exponential utility function. Unlike the method used in [2], we use the functional Ito’s formula to derive the associated HJB equation. We also give a necessary condition for the well-posedness of the HJB equation (see Lemma 3.2).

Stochastic control models with delay have wide range of applications and have been discussed extensively in literatures. Kolmanovaskii and Maizenberg [13] introduced the idea of describing the delay information like (2.1)-(2.2). Elsanousi and Larsson [6] and Larsson and Risserbro [16] have discussed model with finite delay. Elsanousi, Øksendal and Sulem [7] developed and maximum principle for optimal control problem of stochastic systems with delay. The idea of dynamic programming principle for stochastic delay differential equations appears in Gihman and Skorokhod [12] and Kolmanovskii and Shaikhet [17]. Larsson [15] showed dynamic programming principle for the stochastic control problems with delay. In Chang, Pang and Pemy [1], a stochastic control problem with delays in general form was considered and Fréchet derivatives was used to derive the HJB equation in an infinite dimensional space.

2 Problem Formulation

Consider an investor’s portfolio comprises a risky asset and a riskless asset. The riskless asset earns the investor a fixed interest rate \( r > 0 \). Money deposited in the bank can be considered as riskless asset. The consumption of the investor is assumed to come from riskless asset and
we also assume that the investor can freely move her money between the risky and riskless asset at any time.

Assume the process \( \{ B(t), t \geq 0 \} \) is a one-dimensional standard Brownian motion defined on a complete filtered probability space \( (\Omega, \mathcal{F}, P, \mathbf{F}) \), where \( \mathbf{F} = \{ \mathcal{F}^t, t \geq 0 \} \) is the \( \mathcal{F} \)-augmented natural filtration generated by the Brownian motion \( \{ B(t), t \geq 0 \} \).

Let \( K(t) \) be the amount invested in the risky asset and let \( L(t) \) be the amount invested in the riskless asset. Assume \( K(t), L(t) \) satisfy the following stochastic differential equation

\[
dK(t) = \left[ (\mu_1 + \mu_2 Y(t) + \mu_3 Z(t)) K(t) + I(t) \right] dt + \sigma K(t) dB(t),
\]

\[
dL(t) = [rL(t) - C(t) - I(t)] dt,
\]

where \( \mu_1, \mu_2, \mu_3 \) and \( \sigma \) are constants, \( I(t) \) is the investment rate in the risky asset, \( C(t) \) is the consumption rate and the performance of the risky asset depends on \( Y(t) \) and \( Z(t) \), the delay variables, given as

\[
(2.1) \quad Y(t) = \int_{-h}^{0} e^{\lambda \theta} X(t + \theta) d\theta,
\]

\[
(2.2) \quad Z(t) = X(t - h), \quad \forall t \in [s, T],
\]

where \( \lambda > 0 \) is a constant, \( h \) is the delay parameter and \( X(t) \equiv K(t) + L(t) \) is the total wealth. As we can see, \( e^{\lambda \theta} \) serves as a weight function if we regard \( Y(t) \) as the weighted average of the \( X(t + \theta) \), \( \theta \in [-h, 0] \). We assume that \( \lambda > 0 \) because usually the most recent information will be assigned a higher weight so the weight function should be increasing with respect to \( \theta \).

The equation for \( X(t) \) is

\[
(2.3) \quad dX(t) = \left[ (\mu_1 + \mu_2 Y(t) + \mu_3 Z(t)) K(t) + rL(t) - C(t) \right] dt + \sigma K(t) dB(t), \quad \forall t \in [s, T].
\]

The initial condition is the information about \( X(t) \) for \( t \in [-h, s] \):

\[
(2.4) \quad X(t) = \varphi(t - s), \quad \forall t \in [-h, s],
\]

where \( \varphi \in \mathbf{J} \) and \( \mathbf{J} \) is defined by \( \mathbf{J} \equiv C([-h, 0]) \), which is the space for all continuous functions defined on \([ -h, 0]\) equipped with the sup norm

\[
(2.5) \quad ||\varphi|| = \sup_{\theta \in [-h, 0]} |\varphi(\theta)|.
\]

Let \( L^2(\Omega, \mathbf{J}) \) be a Banach space for all \( (\mathbf{F}, \mathbf{B}(\mathbf{J})) \)-measurable maps \( \Omega \to \mathbf{J} \) that are \( L^2 \) in the Bochner sense, where \( \mathbf{B}(\mathbf{J}) \) is the Borel \( \sigma \)-field on \( \mathbf{J} \). For any \( \phi \in L^2(\Omega, \mathbf{J}) \), the Banach norm is given by

\[
(2.6) \quad ||\phi||_2 = \left( \int_{\Omega} ||\phi(\omega)||_2^2 dP(\omega) \right)^{1/2},
\]

where the norm \( || \cdot || \) is given by (2.5).

The state variables are \( X(t) \) and \( Y(t) \). The wealth is allocated between risky and riskless asset and in order to describe it, we treat \( K(t) \) and \( C(t) \) as our control variables. As in [2], we consider the modified model. In the modified model \( X(t) \) satisfies the following equation

\[
(2.7) \quad dX(t) = \left[ (\mu_1 + \mu_2 Y(t) + \mu_3 Z(t)) + rL(t) - C(t) \right] dt + \sigma K(t) dB(t), \quad \forall t \in [s, T].
\]

Instead of using \( K(t) \) and \( C(t) \), we choose \( c(t) = \frac{K(t)}{X(t)} \) and \( k(t) = \frac{C(t)}{X(t)} \) as our consumption and investment controls, respectively. Under certain conditions, we can show that \( X(t) > 0 \) almost surely (see Lemma 2.1), so that the controls \( c(t), k(t) \) are well-defined. More details can be found in [2].

Under the above modifications, the equation for \( X(t) \) can be written as

\[
(2.8) \quad dX(t) = \left[ (\mu_1 - r) k(t) - c(t) + rX(t) \right] dt + \mu_2 Y(t) + \mu_3 Z(t) dt + \sigma k(t) X(t) dB(t), \quad \forall t \in [s, T].
\]

The initial condition is given by

\[
(2.9) \quad X(t) = \varphi(t - s), \quad \forall t \in [s - h, s],
\]

where \( \varphi \in \mathbf{J} \) and \( \varphi(\theta) > 0, \forall \theta \in [-h, 0] \).

For functional differential equations we use the following conventional notation: If \( \psi \in C([-h, T]; \mathbf{R}) \) and \( t \in [0, T] \), let \( \psi_t(\theta) \in \mathbf{J} \) be defined by

\[
(2.10) \quad ||\psi_t(\theta)|| = \sup_{\theta \in [-h, 0]} |\psi_t(\theta)| = \sup_{\theta \in [-h, 0]} |\psi(\theta)|.
\]

Using the above notation, initial condition (2.8) can be written as

\[
(2.11) \quad X_s = \varphi.
\]

Let \( \mathcal{I} \) denote the admissible control space. We assume that a control policy \( (k(t), c(t)) \) in \( \mathcal{I} \) is \( \mathcal{F}^t \)-measurable for any \( t \in [0, T] \) with \( c(t) \geq 0, \forall t \in [0, T] \). Also we assume that

\[
(2.11) \quad \begin{cases} |k(t)X(t)| \leq \Lambda_1 |X(t) + \mu_3 Y(t)|, \\ |c(t)X(t)| \leq \Lambda_2 |X(t) + \mu_3 Y(t)|, \end{cases}
\]

where \( \Lambda_1 > 0, \Lambda_2 > 0 \) are constants.

We have the following result:

**Lemma 2.1.** The solution \( X(t) \) of the system (2.7) - (2.8) satisfies \( X(t) > 0 \) almost surely.

**Proof.** See Lemma 2.2 in [2].

The utility function \( U(C) \) is defined based on the consumption rate. We assume that \( \Psi \) is the terminal utility function and it depends on both \( X(T) \) and \( Y(T) \). The problem under consideration is a portfolio optimization problem on a finite time horizon \([0, T]\) with
the objective function given by

$$J(s, \phi, k, c) = E_{s, \phi} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right], \quad \forall (k, c) \in \Pi.$$  

The value function is given by

$$V(s, \phi) = \sup_{k, c \in \Pi} J(s, \phi, k, c)$$

Note that $V(s, \phi)$ is a functional defined on an infinite dimensional space $[0, T] \times C[-h, 0]$. We turn $V$ into a function defined on a finite dimensional space as follows

$$V(s, \phi) = V(s, x, y, z),$$

where $V : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$, and

$$x = x(\phi) \equiv \varphi(0),$$

$$y = y(\varphi) \equiv \int_0^h e^{\lambda \theta} \varphi(\theta) d\theta.$$

Further, to derive the HJB equation, we will need that the value function $V$ only depends on $(s, x, y, z)$, i.e.

$$V(s, \varphi) = V(s, x, y, z) = V(s, x, y).$$

Actually, this is a necessary condition that we can derive a HJB equation in a finite dimensional space. See Lemma 3.2 for details.

### 3 Functional Ito’s Formula and the HJB Equation.

Recall that we use $X(t)$ to denote the current value and we use $X_t : [-h, 0] \to \mathbb{R}$ to denote the path of $X(t)$ from $t - h$ to $t$. For a functional $f(X_t)$ of $X_t$, we have the following functional Ito’s formula:

$$df(X_t) = \partial_t f(X_t) dt + \partial_x f(X_t) dX(t) + \frac{1}{2} \partial_{xx} f(X_t) d\langle X \rangle(t).$$

where

$$\partial_t f(X_t) = \lim_{\delta \to 0} \frac{f(X_t + \delta) - f(X_t)}{\delta},$$

$$X_t(\theta) = \begin{cases} X_t(\delta + \theta), & \theta \in [-h, -\delta], \\ X_t(0), & \theta \in [-\delta, 0]; \end{cases}$$

$$\partial_x f(X_t) = \lim_{\delta \to 0} \frac{f(X_t^\delta) - f(X_t)}{\delta},$$

$$X_t^\delta(\theta) = \begin{cases} X_t(\theta), & \theta \in [-h, 0), \\ X_t(0) + \delta, & \theta = 0; \end{cases}$$

$$\partial_{xx} f(X_t) = \lim_{\delta \to 0} \frac{\partial_x f(X_t^\delta) - \partial_x f(X_t)}{\delta}.$$  

The above derivatives and the functional Ito’s formula was initiated by Dupire [4] and was later studied in Cont and Fournié [3].

Now let us consider the following functional:

$$y(X_t) = \int_{-h}^0 \phi(\theta) X_t(\theta) d\theta;$$

$$z(X_t) = X_t(-h),$$

where $\phi(\theta)$ is a smooth function with a continuous first order derivative $\phi'(\theta)$.

From the above definition, we can see that the delay variable $y(X_t)$ is actually the weight average of the delayed (historical) value of $X(t + \theta)$ for $\theta \in [0, -h]$ with the weight function given by $\phi(\theta)$, $\theta \in [-h, 0]$. For example, when $\phi(\theta) = e^{\lambda \theta}$, $y(X_t)$ is just the exponential moving average for $X(s)$ for $s \in [t, t - \theta]$. If $\phi(\theta) = 1$, $y(X_t)$ is just the moving average, which is a special case of the exponential moving average (with $\lambda = 0$). As we know, investors tend to look at the moving average or exponential moving average for a stock before they make the investment decision. A stock with current price lower than its exponential moving average may signal the downward trend for the stock price, therefore it will scare away some investors. Due to the weaker demand, the price may go down further. That is why we want to study the system (2.3) with delay variable given by (2.1).

On the other hand, from the following lemma, we will see why we also include the delay variable $z(X_t)$ given by (3.8).

**Lemma 3.1.** If $y(X_t), z(X_t)$ is given by (3.7)-(3.8), then we have

$$dy(X_t) = \left[ X_t(0) \phi(0) - \int_{-h}^0 \phi'(\theta) X_t(\theta) d\theta - \phi(-h)z(X_t) \right] dt.$$  

**Proof.** It is easy to see that $\partial_y y(X_t) = 0$, and $\partial_{xx} y(X_t) = 0$. On the other hand,

$$y(X_t, \delta) = \int_{-h}^{\delta} \phi(\theta) X_t(\delta + \theta) d\theta + \int_{-\delta}^0 \phi(\theta) X_t(0) \phi(0) d\theta.$$  

So we have

$$y(X_t, \delta) - y(X_t)$$

$$= \int_{-h+\delta}^{\delta} \phi(\theta) X_t(\delta) d\theta + \int_{-\delta}^{0} \phi(\theta) X_t(0) d\theta$$

$$= \int_{-h}^{\delta} \phi(\theta) X_t(\theta) d\theta - \int_{-h}^{\delta} \phi(\theta) X_t(\theta) d\theta.$$
Assume that the value function $V$ is given by (2.7)-(2.8). Then we have

$$V(s, x, y) = \sup_{(k, c) \in \Pi} E_{s, \varphi, k, c} \left[ \int_{t}^{T} e^{-\beta(t-s)} U(c(t)) X(t) \, dt + e^{-\beta(t-s)} V(t, X(t), Y(t)) \right],$$

for all $\mathcal{F}^t$-stopping time $t \in [s, T]$ and $(x, y) \in \mathbb{R}^2$, where $\varphi \in \mathcal{J}$ is such that $x = x(\varphi) = X(s)$ and $y = y(\varphi) = Y(s)$.

**Proof.** The proof is very similar to Theorem 5.1 of [15] so it is omitted here.

By virtue of the above lemma and Lemma 3.1, it is easy to derive the HJB equation. The result is given in the following theorem.

**Theorem 3.1.** (HJB Equation) Assume that (3.10) holds and $V(s, x, y) \in C^{1,2}[\{0\} \times \mathbb{R} \times \mathbb{R}]$. Then the value function $V(s, x, y)$ given by (2.13) and (3.10) satisfies the following HJB equation

$$(3.11) \max_{k, c \geq 0} \mathcal{L}^{k,c} V(s, x, y) + U(cx) - \beta V(s, x, y) + (x - \lambda y - e^{-\lambda h} z)V_y(s, x, y) = 0, \quad \forall z \in \mathbb{R},$$

where $\mathcal{L}^{k,c}$ is defined by

$$(3.12) \mathcal{L}^{k,c} V(s, x, y) \equiv V_x + \frac{1}{2} \sigma^2 k^2 x^2 V_{xx} + ((\mu - r) k - c + r)x + \mu_2 y + \mu_3 z)V_z,$$

and the boundary condition is

$$(3.13) V(T, x, y) = \Psi(x, y).$$

The proof is similar to Theorem 5.1 in [15] so we omit the proof here.

In [2], the results for HARA utility function $U(C) = \frac{1}{\gamma} C^\gamma$, $\gamma < 1$, $\gamma \neq 0$ were given. In this paper, we obtain the results for logarithmic utility function and exponential utility function.

4 **Logarithmic Utility Function.**

In this section we consider the logarithmic utility function given by

$$U(x) = \log x.$$

Now the HJB equation is

$$(4.1) \max_{k, c \geq 0} \mathcal{L}^{k,c} V(s, x, y) + \log(cx) - \beta V(s, x, y) + (x - \lambda y - e^{-\lambda h} z)V_y(s, x, y) = 0,$$

where $\mathcal{L}^{k,c}$ is given by (3.12). The candidates for optimal controls are

$$k^* = -\frac{(\mu_2 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{x V_x}.$$
We look for the solution of the form

\((4.3)\quad V(s, x, y) = \psi(x, y) + Q(s),\)

where \(Q(s)\) and \(\psi(x, y)\) will be determined. Moreover, we have

\[
V_x = \psi_x(x, y), \quad V_{xx} = \psi_{xx}(x, y), \\
V_y = \psi_y(x, y), \quad V_s = Q'(s).
\]

Plugging \(k^*, c^*\) and above equations into (4.2), we obtain

\((4.4)\quad \beta[\psi(x, y) + Q(s)] - Q'(s) = -\frac{(\mu_1 - r)^2}{2\sigma^2}\psi_x^2 + \log \frac{1}{\psi_x} \quad 1 + \frac{1}{\beta}(r + \mu_3 e^{\lambda h}).\)

Define

\((4.5)\quad u \equiv x + \mu_3 e^{\lambda h}y,\)

and let \(\psi(x, y) = \frac{1}{\lambda} \log(u).\) Then we have,

\[
\psi_x = \frac{1}{\beta u}, \quad \psi_{xx} = -\frac{1}{\beta u^2}, \quad \psi_y = \frac{\mu_3 e^{\lambda h}}{\beta u}.
\]

Assume that

\((4.6)\quad \mu_3 e^{\lambda h}(r + \mu_3 e^{\lambda h}) = \mu_2 - \mu_3 \lambda e^{\lambda h},\)

By virtue of assumption (4.6) and definition of \(u\), equation (4.4) can be written as

\((4.7)\quad Q'(s) = \beta Q(s) - \frac{(\mu_1 - r)^2}{2\beta\sigma^2} - \log \beta - 1 + \frac{1}{\beta}(r + \mu_3 e^{\lambda h}).\)

Let

\((4.8)\quad \Lambda_4 = \frac{(\mu_1 - r)^2}{2\beta\sigma^2} + \log \beta - 1 + \frac{1}{\beta}(r + \mu_3 e^{\lambda h}).\)

Equation (4.7) can be written as

\((4.9)\quad \frac{d}{ds}(\log(\beta)Q(s)) = -\Lambda_4 e^{-\beta s}.\)

At terminal time \(t = T\), we have

\[
V(T, x, y) = Q(T) + \frac{1}{\beta} \log(x + \mu_3 e^{\lambda h}y).
\]

The terminal utility function \(\Psi(x, y)\) is assumed to be consistent with the log utility function. In particular, for technical reasons, we assume that the function \(\Psi(x, y)\) is of the form

\((4.10)\quad \Psi(x, y) = \frac{1}{\beta} \log(x + \mu_3 e^{\lambda h}y).\)

Then, by virtue of (4.10) and (4.10), we can get the boundary condition for \(Q(s)\) at \(s = T\)

\((4.11)\quad Q(T) = 0.\)

The solution for (4.9)-(4.11) is given as

\((4.12)\quad Q(s) = \frac{\Lambda_4}{\beta} (1 - e^{-\beta(T-s)}).\)

If \(\Lambda_4 > 0\), we have

\((4.13)\quad Q(s) \geq 0, \quad \forall s \in [0, T].\)

Therefore, the equations (3.11)-(3.13) have the solution

\((4.14)\quad V(s, x, y) = Q(s) + \log(x + \mu_3 e^{\lambda h}y),\)

Moreover, the candidate optimal investment and consumption rates are given as

\((4.15)\quad k(s)^* = \frac{(\mu_1 - r)(x + \mu_3 e^{\lambda h}y)}{\sigma^2},\)

\((4.16)\quad c(s)^* = \frac{\beta(x + \mu_3 e^{\lambda h}y)}{x},\)

where \(Q(s)\) is given by (4.12), and \(x\) and \(y\) are estimated at time \(s\) as following

\[
x = X(s), \quad y = Y(s) = \int_{-h}^{s} e^{\lambda h}X(s + \theta)d\theta.
\]

To ensure that the solution given by (4.14) is equal to the value function (2.13), we need the following verification theorem.

**Theorem 4.1. (Verification Theorem)** Assume that \(X(t)\) be a strong solution of (2.7)-(2.8) and \(Y(t)\) and \(Z(t)\) are given by (2.1) and (2.2) respectively. Let \(V(s, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})\) be a solution of the HJB equation given by (3.11)-(3.13) such that

\((4.17)\quad \mathbf{E} \left[ \int_0^T [k(t) X(t) V_x(t, X(t), Y(t))]^2 dt \right] < \infty, \quad \forall(k, c) \in \Pi.\)

Then we have,

\[
V(s, x, y) \geq J(s, \varphi, k, c), \quad \forall(k, c) \in \Pi
\]

where \(J(\cdot)\) is given by (2.12). In addition, assume that the utility function is given by

\((4.18)\quad U(x) = \log x\)

and

\[
k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2V_{xx}}, \quad c^* = \frac{1}{xV_x}.
\]

If \((k^*, c^*) \in \Pi,\) then \((k^*, c^*)\) is the optimal control policy. In this case, we have

\[
V(s, x, y) = J(s, \varphi, k^*, c^*).
\]

The proof of this theorem is very similar to the proof of Theorem 4.1 in Chang, Pang, and Yang [2], so we omit it here. Now it remains to verify that the function defined by (4.14) is a classical solution of (3.11)-(3.13) and the optimal control policy is given by (4.15)-(4.16)

**Theorem 4.2.** Assume that \(X(t)\) is a strong solution of (2.7)-(2.8) and \(Y(t)\), \(Z(t)\) are given by (2.1) and
(2.2). Assume that the utility function is given by
\[ U(x) = \log x \] and that the terminal function is given by
\[ \Psi(x, y) = \frac{1}{3} \log (x + \mu_3 e^{\lambda h} y). \] Suppose (4.6) also holds. Then the function \( V(s, x, y) \) given by (4.14) is a classical solution of the HJB equation (3.11)-(3.13), and it is equal to the value function defined by (2.13)-(2.17), that is
\[
V(s, x, y) = \sup_{(k, c) \in \Pi} J(s, \varphi, k, c).
\]
In addition, the optimal control policy \((k^*, c^*)\) is given by (4.15) and (4.16).

Proof. It is evident from the derivation of \( V(s, x, y) \) that \( V(s, x, y) \) given by (4.14) is a classical solution of the HJB equation (3.11)-(3.13).

To use Theorem 4.1, we first verify that condition (4.17) is satisfied. Using (4.14), we can get
\[
|V_z(t, X(t), Y(t))| = \left| \frac{1}{X(t) + \mu_3 e^{\lambda h} Y(t)} \right|
\]
Using the definition of admissible control space \( \Pi \), we have
\[
|k^*(t)X(t)| \leq \Lambda_1 |X(t) + \mu_3 Y(t)|
\]
\[
\leq \Lambda_1 |X(t) + \mu_3 e^{\lambda h} Y(t)| = \Lambda_1.
\]
Therefore, we have
\[
|k^*(t)X(t)V_z(t, X(t), Y(t))| \leq \Lambda_1 |X(t) + \mu_3 e^{\lambda h} Y(t)| \cdot \left| \frac{1}{X(t) + \mu_3 e^{\lambda h} Y(t)} \right| = \Lambda_1.
\]
We have,
\[
E \left[ \int_0^T |k^*(t)X(t)V_z(t, X(t), Y(t))|^2 dt \right] \leq E \left[ \int_0^T \Lambda_1^2 dt \right] = \Lambda_1^2 T < \infty.
\]
where \( \Lambda_1 > 0 \) is a constant independent of \( t \). Thus condition (4.17) is verified. Using definition of \((k^*, c^*)\), it is easy to see that \((k^*(t), c^*(t))\) is \( \mathcal{F}^t \)-measurable for any \( t \in [0, T] \). Also, by virtue of (4.16), it is easy to get that \( c^*(t) \geq 0, \forall t \in [0, T] \). The conditions given by (2.11) of admissible control space are also satisfied. Hence we get \((k^*, c^*) \in \Pi\). This completes the proof.

5 Exponential Utility Function.

In this section we consider the exponential utility function given as 
\[ U(x) = 1 - e^{-\alpha x}, \quad \alpha > 0. \] We note that maximizing the utility function with or without the additive term 1 gives the same results. The additive term 1 in the utility function restricts the range of the function between 0 and 1 and other than that it does not have any mathematical relevance. So we drop the term 1 for technical convenience and consider the following
\[
U(x) = -e^{-\alpha x}, \quad \alpha > 0.
\]
Now the HJB equation is
\[
\max_{k,c \geq 0} \left\{ \mathcal{L}^{k,c}V(s, x, y) - e^{-\alpha c} \right\} \beta V(s, x, y) + (x - \lambda y - e^{-\lambda h} z)V(s, x, y) = 0,
\]
where \( \mathcal{L}^{k,c} \) is given by (3.12). The candidates for optimal controls are
\[
k^* = \frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = -\frac{1}{\alpha} \log \frac{V_x}{\alpha}.
\]
We have the following verification theorem.

Theorem 5.1. (Verification Theorem) Assume that \( X(t) \) is a strong solution of (2.7)-(2.8) and \( Y(t) \) and \( Z(t) \) are given by (2.1) and (2.2) respectively. Let \( V(s, x, y) \in C^{1,2}([0, T] \times \mathbb{R} \times \mathbb{R}) \) is a solution of the HJB equation given by (3.11)-(3.13) such that (4.17) holds. Then we have,
\[
V(s, x, y) \geq J(s, \varphi, k, c), \quad \forall (k, c) \in \Pi
\]
where \( J(\cdot) \) is given by (2.12). In addition, assume that the utility function is given by \( U(x) = -e^{-\alpha x} \) and \((k^*, c^*)\) are given by (5.3)

If \((k^*, c^*) \in \Pi\), then \((k^*, c^*)\) is the optimal control policy. In this case, we have
\[
V(s, x, y) = J(s, \varphi, k^*, c^*).
\]
The proof of this theorem is very similar to the proof of Theorem 4.1 in Chang, Pang and Yang [2], so we omit it here. Now we need to find a classical solution of (3.11)-(3.13) and derive the optimal control policy.

We look for the solution of the form
\[
V(s, x, y) = \psi(x, y) Q(s).
\]
Then we have
\[
V_x = Q(s) \psi_x(x, y), \quad V_{xx} = Q(s) \psi_{xx}(x, y), \quad V_y = Q(s) \psi_y(x, y), \quad V_s = Q'(s) \psi(x, y).
\]
Substituting above into equation (5.2) yields
\[
\beta[\psi Q(s)] - \psi Q'(s) = -\frac{(\mu_1 - r)^2 Q(s) \psi_x^2}{2 \sigma^2 \psi_{xx}} + \frac{Q(s) \psi_x}{\alpha} \left[ \log \left( \frac{Q(s) \psi_y}{\alpha} \right) - 1 \right] + (r x + \mu_2 y + \mu_3 z) Q(s) \psi_x + (x - \lambda y - e^{-\lambda h} z) Q(s) \psi_y.
\]
Let
\[
u \equiv x + \mu_3 e^{\lambda h} y,
\]
and
\[
\psi(x, y) = -e^{-\gamma u},
\]
Let $γ$ be a constant to be determined. Now we can get
\[
ψ_x = -γψ, \quad ψ_{xx} = -γ^2ψ, \quad ψ_y = -γμ_3e^{λh}ψ.
\]
Using the above substitutions in equation (5.2), we can obtain
\[
ψ[βQ(s) − Q′(s)] = \frac{(μ_1 - r)^2 Q(s)ψ}{2σ^2}
+ \frac{Q(s)}{α}(-γψ) \left[ \log\left(\frac{γψQ(s)}{α}\right) - 1 \right]
− γ(rx + μ_3y)Q(s)ψ − γ(x − λy)Q(s)μ_3e^{λh}ψ.
\]
Canceling $ψ$ and noting that $\log(-ψ) = log e^{-γu} = -γu$, we can rewrite the above equation to
\[
−βQ(s) + Q′(s) = \frac{(μ_1 - r)^2 Q(s)}{2σ^2}
+ \frac{γQ(s)}{α} \log\left(\frac{γQ(s)}{α}\right) − γu - 1
+ [(r + μ_3e^{λh})x + (μ_2 - λy)μ_3e^{λh}] γQ(s).
\]
Now let us assume that
\[
μ_3e^{λh}(r + μ_3e^{λh}) = μ_2 - μ_3 λe^{λh},
\]
and
\[
γ = α(r + μ_3e^{λh}).
\]
Then by assumption (5.5), we can cancel $u$ from the equation and obtain
\[
−βQ(s) + Q′(s) = \frac{(μ_1 - r)^2 Q(s)}{2σ^2}
+ [(r + μ_3e^{λh})Q(s) [log(r + μ_3e^{λh}) + log Q(s) - 1].
\]
The above equation can be rewritten as
\[
\frac{Q′(s)}{Q(s)} = β + \frac{(μ_1 - r)^2}{2σ^2} − (r + μ_3e^{λh})
+ (r + μ_3e^{λh}) log ((r + μ_3e^{λh})Q(s)).
\]
Let
\[
Λ_5 = β + \frac{(μ_1 - r)^2}{2σ^2} − (r + μ_3e^{λh})[log(r + μ_3e^{λh}) − 1].
\]
Equation (5.8) can be written as
\[
\frac{Q′(s)}{Q(s)} = Λ_5 + (r + μ_3e^{λh}) log Q(s).
\]
At the terminal time $t = T$, we have
\[
V(T,x,y) = Q(T)(e^{−γu}) = Ψ(x,y)
\]
The terminal utility function $Ψ(x,y)$ is assumed to be consistent with the exponential utility function. In particular, we assume that it is of the form
\[
Ψ(x,y) = −Λ e^{−γ(x+μ_3e^{λh})y} = −Λ e^{−γu},
\]
where $γ$ is given by (5.7). So the boundary condition for $Q(s)$ at $s = T$ is
\[
Q(T) = Λ.
\]
The explicit solution for (5.10)-(5.12) is given as
\[
Q(s) = \exp\left(\frac{Λ_5}{r + μ_3e^{λh}} \left(e^{−(r+μ_3e^{λh})(T-s)} - 1\right) + e^{−(r+μ_3e^{λh})(T-s)} \log Λ\right).
\]
It is easy to see that $Q(s) > 0$ and $Q(s)$ is an increasing function for $s \in [0,T]$. Therefore, we can get
\[
0 < Q(s) < Λ \quad ∀s \in [0,T].
\]
The HJB equations (3.11)-(3.13) have the solution
\[
V(s,x,y) = −Q(s)e^{−α(r+μ_3e^{λh})(x+μ_3e^{λh}y)},
\]
The optimal investment and consumption rates are given as
\[
k^∗(s) = \frac{(μ_1 - r)}{α(r + μ_3e^{λh})σ^2x},
\]
\[
c^∗(s) = \frac{1}{xα} \left[\log((r + μ_3e^{λh})Q(s))
− α(r + μ_3e^{λh})(x + μ_3e^{λh}y)\right],
\]
where $Q(s)$ is given by (5.13), and $x$ and $y$ are estimated at time $s$ as following
\[
x = X(s), \quad y = Y(s) = \int_{−h}^{s} e^{λh}X(s+θ)dθ.
\]
By virtue of (5.14) and Lemma 2.1, we can see that $c^∗(s) ≥ 0$ as long as
\[
Λ(r + μ_3e^{λh}) ≤ 1.
\]
Now we can prove the following verification theorem.

**Theorem 5.2.** Assume that the utility function is given by (5.1) and the terminal function is given by (5.11). In addition, Suppose (5.6) and (5.7) also hold. Then the function $V(s,x,y)$ given by (5.15) is a classical solution of the HJB equation (3.11)-(3.13), and it is equal to the value function defined by (2.13)-(2.17), that is
\[
V(s,x,y) = \sup_{(k,c) ∈ Π} J(s,φ,k,c).
\]
In addition, assume that (5.18) holds. Then the optimal control policy $(k^∗,c^∗)$ is given by (5.16) and (5.17).

**Proof.** From the derivation of $V(s,x,y)$, it is easy to check that $V(s,x,y)$ given by (5.15) is a classical solution of HJB equation (3.11)-(3.13).

To use Theorem 5.1, we first verify that condition (4.17) is satisfied. Using (5.15) and
\[
V_x(t,X(t),Y(t)) = γQ(t)e^{−γ(X(t)+μ_3e^{λh}Y(t))},
\]
where $\gamma > 0$ is a constant given by (5.7). In addition, it is easy to verify that

$$0 < \frac{1 + \gamma x}{e^{\gamma x}} \leq 1, \quad \forall x > 0.$$  

So we can get

$$\left| e^{-\gamma(X(t)+\mu_3e^{\beta Y(t)})} \right| \leq \frac{1}{1 + \gamma(X(t)+\mu_3e^{\beta Y(t)})},$$

for all $t \in [s,T]$. Therefore, by virtue of (5.14), we can get

$$|V_k(t,X(t),Y(t))| < \frac{\gamma \Lambda}{1 + \gamma(X(t)+\mu_3e^{\beta Y(t)})},$$

Using the definition of admissible control space II, we have

$$|k^+(t)X(t)| \leq \Lambda_1|X(t)+\mu_3Y(t)| \leq \Lambda_1|X(t)+\mu_3e^{\beta Y(t)}|.$$  

Therefore, we can get

$$|k^+(t)X(t)V_k(t,X(t),Y(t))| \leq \Lambda_1 \left| \frac{\gamma \Lambda(X(t)+\mu_3e^{\beta Y(t)})}{1 + \gamma(X(t)+\mu_3e^{\beta Y(t)})} \right|$$

$$\leq \Lambda_1 \Lambda \left( 1 + \frac{1}{1 + \gamma(X(t)+\mu_3e^{\beta Y(t)})} \right)$$

$$\leq 2\Lambda_1 \Lambda,$$

where $\Lambda_1, \Lambda$ are positive constants independent of $t$. Now we can get,

$$\mathbb{E} \left[ \int_0^T \left| k^+(t)X(t)V_k(t,X(t),Y(t)) \right|^2 dt \right]$$

$$\leq \mathbb{E} \left[ \int_0^T 4\Lambda_1^2 \Lambda^2 dt \right] = 4\Lambda_1^2 \Lambda^2 T < \infty.$$  

Thus condition (4.17) is verified. By definition of $(k^*, c^*)$, it is easy to see that $(k^*(t), c^*(t))$ is $\mathcal{F}^t$-measurable for any $t \in [0,T]$. Also, by virtue of (5.14), it is easy to get that $c^*(t) \geq 0, \forall t \in [0,T]$. The conditions given by (2.11) of admissible control space are also satisfied. Hence we have $(k^*, c^*) \in \Pi$. This completes the proof.

References


