On the Correlation Approach and Parametric Approach for CVA Calculation

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Abstract Credit value adjustment (CVA) is an adjustment added to the fair value of an over-the-counter trade due to the risk of counterparty defaults. When the exposure to the counterparty and the counterparty default risk tend to change in the same direction, the so-called wrong-way risk (WWR) must be taken into account. The right-way risk (RWR) takes place when the two factors move in the opposite directions. These two effects are also called directional-way risk (DWR). A lot of efforts have been made to reduce the computational burden to calculate CVA with DWR. The two most popular approaches are the parametric approach and the correlation approach. In this paper we develop a connection between these two approaches. In particular, by decomposing the DWR into a robust correlation coefficient and a profile multiplier, we bring the parametric approach into the correlation approach framework. This study allows us to explain the parameters in the parametric approach. Our results also suggest that the parametric approach can become sensitive when calculating the WWR in certain scenarios. For risk model governance and validation purposes, cautions should be given while using the parametric approach for CVA calculation.

Keywords: Credit Value Adjustment (CVA), Wrong Way Risk, Right Way Risk, Correlation Approach, Parametric Approach.
1 Introduction

Over-the-counter derivative dealers face the risk of defaults by the counterparties. The counterparty default can induce significant loss when a dealer has positive exposure to the counterparty. Credit value adjustment, known as CVA, is thus introduced to capture the actual fair value to the dealer. It is a value on top of the market value of the derivative itself. That means that the fair value of an over-the-counter derivative should depend on the credit quality of the counterparty as well as the value of the derivative itself. After the financial crisis in 2008, CVA has gained more attention for both fair valuation and capital requirement purposes. CVA is usually quoted as a part of over-the-counter derivatives. Basel III, Basel Committee on Banking Supervision (BCBS) (2011), has also enhanced its requirement on the CVA capital charge.

There are mainly two risk sources in CVA. One is the collateralized and netted exposure at default with the counterparty. The other is the default probability of the counterparty. For simplicity, in the past, the over-the-counter derivatives used to be priced with CVA assuming independence between the exposure and the counterparty credit risk. However, in reality, these two risk sources often move at the same time. A wrong-way risk (WWR) intensifies counterparty risk when the dealer’s exposure to the counterparty tends to increase while the counterparty credit quality deteriorates. A famous example of WWR is the CDS (credit default swap) contracts during the recent financial crisis where the credit protection buyers had to face the significant credit risk from major CDS sellers like AIG while the underlying protection got more valuable. On the other hand the right-way risk (RWR) is also plausible when the two factors move in the opposite directions. That is, when the counterparty credit risk intensifies the exposure to the counterparty goes down or vice versa. RWR essentially builds in a hedge between the underlying derivative risk and the counterparty risk. Because of this hedging effect, RWR gets less attention from the regulators. For example Basel III does not offer any incentive rewarding RWR as much as it punishes the WWR. These two co-movement effects (WWR and RWR) are often called directional-way risk (DWR).

According to Gregory (2012), there are mainly two types of approaches to model CVA with DWR, the correlation approach and the parametric approach. The first approach attempts to capture the underlying risk through the correlation between the exposure and the counterparty credit risk. In most cases, the solution to this approach requires simulations. Pykhtin and Rosen (2010) considers the correlation structure of portfolio exposures and idiosyncratic default risk driver and derives analytical results based on the joint normal distribution. Rosen and Saunders (2012) uses similar assumptions about portfolio exposures and idiosyncratic default risk driver and proposes an algorithm to incorporate different default scenarios with pre-generated exposures to model WWR. Both Brigo and Alfonsi (2005) and Brigo and Pallavicini (2008) exam-
ine the relation between counterparty default risk and market observed CDS and then price counterparty risk based on the stochastic process of CDS. Skoglund, Vestal and Chen (2013) and Pang, Chen and Li (2015) use joint stochastic processes to model the risk drivers of the underlying and the CDS. In the latter one, the authors propose a robust correlation method to model the correlation of the underlying processes for exposures and default probabilities.

On the other hand, the parametric approach can be viewed as a reduced form approach where the model does not necessarily explain the underlying correlation but only aims to capture the co-movement itself. The examples of this approach include Hull and White (2012), Ghamami and Goldberg (2014) and Ruiz, Boca and Pachón (2015). Hull and White (2012) assumes the hazard rate of the default follows an exponential function of the exposure and the WWR is modeled by a single parameter. Ghamami and Goldberg (2014) assumes the exposure is a function of a uniform random variable. Using the Hull and White model, they find that independent CVA can exceed the wrong way CVA under some circumstances. Ruiz, Boca and Pachón (2015) carries out calibrations for some parametric models with different function forms.

In this paper, we first establish a connection between the parametric approach and the correlation approach. This connection allows us to discover economic interpretations of the parameters in the latter approach. We apply the DWR decomposition approach (referred as the PCL Approach in this paper) introduced by Pang, Chen and Li (2015) to the parametric approach (referred as the Hull-White Approach) proposed in Hull and White (2012). As a result, we derive a decomposition and an analytical and economic anatomy of the parameter $b$ (see equation (3)) in the Hull-White Approach. In addition our results provide an explanation to the finding of the effect of the underlying price volatility on the CVA value observed by Ruiz, Boca and Pachón (2015). The parameter $b$ in the Hull-White Approach blends both the robust correlation and the profile multiplier in the PCL Approach. CVA value can be quite sensitive when $b$ is small. This is very important consideration for model risk in real implementation.

The rest of the paper is organized as follows. First in Section 2, we briefly review the CVA and the DWR. In Section 3, we introduce the Hull-White parametric approach proposed in Hull and White (2012). Then in Section 4, we briefly review the PCL Approach and DWR decomposition proposed in Pang, Chen and Li (2015). The key analysis results are given and discussed in Section 5. In Section 6, some discussions on the major findings are given. A numerical study based on interest rate swaps is provided in Section 7 with further discussions. At the end we conclude in Section 8.
2 The CVA Calculation

CVA is an adjustment added to the fair value of an over-the-counter trade due to the counterparty credit risk. For simplicity, throughout the paper we consider unilateral CVA only. A general expression for CVA is

\[ CVA = (1 - R)E \left[ 1_{\{\tau \leq T\}} V^+(\tau) \right], \tag{1} \]

where \( R \) is the constant recovery rate, \( \tau \) is the time to default, \( T \) is the time to expiration, \( 1_{\{\tau \leq T\}} \) is the default indicator function that is 1 if the default time \( \tau \leq T \) and 0 otherwise, and \( V^+(t) \) is the risk-neutral discounted exposure at time \( t \) subject to netting and collateral.

Equation (1) can also be expressed as

\[ CVA = (1 - R) \int_0^T E \left[ V^+(t) \right] dF(t), \tag{2} \]

where \( F(t) \) is the cumulative distribution function (CDF) of the counterparty’s time to default \( \tau \).

If the default probability \( F(t) = Pr(\tau \leq t) \) is independent of the credit exposure \( V(t) \), the CVA calculation is relatively easy. However, in reality, the credit exposure \( V(t) \) and the counterparty default probability often depend on each other. When the credit exposure and the default probability tend to move in the same direction, the loss due to the counterparty default would be amplified. In this case, a wrong way risk (WWR) is presented. On the other hand, if the credit exposure and the default probability tend to move in opposite directions, the loss due to the counterparty default would be mitigated. In this case, a right way risk (RWR) is presented. WWR and RWR are often called the directional way risk (DWR).

The presence of the DWR makes the CVA calculation more challenging, as the dependence structure between the credit exposure and the counterparty default risk is not easy to be characterized quantitatively. Next, we will introduce two approaches: the Hull-White parametric approach proposed in Hull-White (2012) and the PCL correlation approach proposed in Pang, Chen and Li (2015).

3 The Hull-White Parametric Approach

Hull and White (2012) proposes a parametric approach to model the dependence of the default probability to the credit exposure. They allow the evolution of the conditional default probability to explicitly depend on the process of the counterparty
credit exposure $V(t)$ at time $t$. In particular, they assume that the hazard rate $h(t)$ of the counterparty default time is an exponential function of the credit exposure $V(t)$:

$$h(t) = \exp(a(t) + bV(t)), \quad (3)$$

where $a(t)$ is a function of time $t$ and $b$ is a constant that describes the dependence of the default probability to the credit exposure. Here we give an outline of the approach and more details can be found in Hull and White (2012).

Conditional on no earlier default by time $t$, the probability of default in any small increment $\Delta t$ is thus $h(t)\Delta t$ where $h(t)$ is a hazard function. At time 0, the probability of default by time $t > 0$ is thus

$$F(t) = 1 - \exp\left( - \int_0^t h(u) du \right). \quad (4)$$

The corresponding discrete form is

$$F(t) = 1 - \exp\left( - \sum_{t_j \leq t} h(t_j) \Delta t \right). \quad (5)$$

Hence the default probability in terms of hazard rates is

$$q(t_i) = \exp\left( - \sum_{k=1}^{i-1} h(t_k) \Delta t \right) - \exp\left( - \sum_{k=1}^i h(t_k) \Delta t \right). \quad (6)$$

Hazard rates are not directly observable from the market. But the counterparty credit spread $s(t)$ is observable. The connection between the hazard rate function $h(t)$ and the credit spread $s(t)$ is given by

$$\exp\left( - \int_0^t h(u) du \right) = \exp\left( \frac{-s(t)t}{1-R} \right), \quad (7)$$

where $s(t)$ is the counterparty credit spread with maturity $t$ assuming constant recovery rate $R$. For a stochastic hazard rate, the following relationship must be satisfied

$$\mathbb{E} \left[ \exp\left( - \int_0^t h(u) du \right) \right] = \exp\left( \frac{-s(t)t}{1-R} \right). \quad (8)$$

The corresponding discrete version is

$$\mathbb{E} \left[ \exp\left( - \sum_{i=1}^{j} h_i \Delta t \right) \right] = \exp\left( \frac{-s_j t_j}{1-R} \right), \quad (9)$$
where $h_i$ and $s_j$ are $h(t_i)$ and $s(t_j)$ respectively, and $t_i, i = 0, 1, 2, \ldots$ are the discrete times corresponding to the different maturities of the counterparty credit spread.

Based on (3), the discrete version of the Hull-White model is

$$h_i = \exp (a(t_i) + bV(t_i)),$$

where $b$ is a constant that measures the amount of DWR, $a(t_i)$ is a function of $t_i$. The values of $a(t_i)$ and $b$ can be calibrated with market data using equations (9) and (10).

In the Hull-White Approach, the function $a(t)$ models the time dependence, and the parameter $b$ is models the co-movement of the credit exposure and the counterparty default probability. Intuitively, the larger the value of $b$, the more dependence between the portfolio value and the counterparty default probability. However, whether the dependence comes from the correlation, the exposure profile, or both is not clear. This is the myth we try to solve in this paper.

### 4 The PCL Correlation Approach

Pang, Chen and Li (2015) proposes a correlation approach. They use a semi-parametric method to decompose the CVA dependence into a robust correlation coefficient and a profile multiplier. It turns out that the multiplier only depends on the first two moments of the credit exposures and the default probabilities, so the dependence is reflected in the robust correlation coefficient. Next we will briefly introduce the PCL Approach.

By taking discretization of the equation (2) into $t_i$ time steps where $t_0 = 0$ and $t_K = T$ and denoting

$$q(t_i) = F(t_i) - F(t_{i-1})$$

as the probability of default at $(t_{i-1}, t_i]$, equation (2) can be rewritten as

$$CVA = (1 - R) \sum_{i=1}^{K} \mathbb{E} [V^+(t_i)q(t_i)].$$

When the underlying exposure and counterparty credit risk are independent, equation (12) becomes

$$CVA_{IND} = (1 - R) \sum_{i=1}^{K} \mathbb{E} [V^+(t_i)] \mathbb{E} [q(t_i)].$$

Pang, Chen and Li (2015) proposes a CVA DWR multiplier decomposition using simple correlation between the exposure and the counterparty risk. Moreover, it is
showed that the DWR CVA can be expressed as a factor adjustment to the CVA under the independence assumption:

\[
CVA_{DWR} = (1 + \bar{\rho}C_p)CVA_{IND},
\]

where

\[
\bar{\rho} = \frac{\sum_{j=1}^{K} \rho(t_j)\sigma_V(t_j)\sigma_{PD}(t_j)}{\sum_{j=1}^{K} \sigma_V(t_j)\sigma_{PD}(t_j)}
\]

is called the robust correlation coefficient and

\[
C_p = \frac{\sum_{j=1}^{K} \sigma_V(t_j)\sigma_{PD}(t_j)}{\sum_{j=1}^{K} \mu_V(t_j)\mu_{PD}(t_j)}
\]

is called the profile multiplier. In the above two equations, \(\mu_V(t_j), \mu_{PD}(t_j), \sigma_V(t_j)\) and \(\sigma_{PD}(t_j)\) are the means and standard deviations of the underlying exposure and the counterparty’s default probability at time \(t_j\) respectively; \(\rho(t_j)\) is the correlation between the exposure and the counterparty’s default probability at time \(t_j\). It is easy to see that \(-1 \leq \bar{\rho} \leq 1\). \(C_p\) describes a quasi coefficient of variation of the defaultable value, that is, the product of default-free value and default probability.

Based on (14), we can define the CVA ratio as

\[
CVA_{ratio} \equiv \frac{CVA_{DWR}}{CVA_{IND}} = 1 + \bar{\rho}C_p.
\]

If there is no DWR, the ratio is 1. For WWR, the ratio will be larger than 1 and for RWR, the ratio is less than 1. In Basel III (2011), the suggested value for this ratio is 1.4 for WWR.

Under this PCL Approach framework, the profile multiplier \(C_p\) does not change with the correlation level and can be derived from the case of independent exposure and default. The robust correlation coefficient \(\bar{\rho}\) can be calibrated as a function of the underlying correlation which correlates the stochastic processes that drive exposure and default. Pang, Chen and Li (2015) also discussed the stability and reusability of these two measures (\(\bar{\rho}\) and \(C_p\)) at certain confidence levels with a series of numerical studies. They propose an efficient algorithm to compute CVA with DWR based on these two measures. A major advantage of the PCL Approach is that a risk manager can infer the confidence interval of CVA price estimate based on the confidence interval of the underlying correlation without adding significant computational burden.
5 Analytical Anatomy

In this section, we derive some analytic results first. The analysis in this section is based on the following assumptions and notations:

(1.) The market is observable daily and there are $K$ time periods;

(2.) $\Delta t = \frac{1}{252}$ and $t_j = j \Delta t$ for $j = 1, 2, \ldots, K$;

(3.) The profit and loss at $t_j$ is denoted as $X_j$;

(4.) $X_j$ follows normal distributions with mean $\mu_j$ and variance $\sigma_j^2$ and $X_i$ and $X_j$ are independent for $i \neq j$;

(5.) The initial value of the portfolio is a positive constant $V_0$;

(6.) The observed credit spread with maturity $t_j$ is $s_j$;

(7.) The recovery rate and the discount rate are 0;

(8.) The exposure at time $t_j$ is $V_j$ and $V_j = V_0 + \sum_{i=1}^j X_i$.

We define the hazard rate at time $t_j$ as $h_j$. Following Hull and White (2012), $h_j$ is a function of $V_j$ and

$$h_j = g(V_j) = \exp(a_j + bV_j),$$

(18)

where $b$ is a pre-determined constant and $a_i$ is time dependent and should be calibrated with the market data. The market implied probability of default between time $t_{j-1}$ and $t_j$, denoted as $C_j$, is

$$C_j \equiv \exp(-s_{j-1}t_{j-1}) - \exp(-s_jt_j).$$

(19)

We define the means and variances of the exposures and default probabilities as follows

$$\mathbb{E}[V_j] = \mu_V(t_j), \quad \text{Var}(V_j) = \sigma_V^2(t_j),$$

$$\mathbb{E}[h_j \Delta t] = \mu_{PD}(t_j), \quad \text{Var}(h_j \Delta t) = \sigma_{PD}^2(t_j).$$

Therefore, we can get

$$\mu_V(t_j) = V_0 + \sum_{i=1}^j \mu_i, \quad \sigma_V^2(t_j) = \sum_{i=1}^j \sigma_i^2,$$

(20)

$$\mu_{PD}(t_j) = C_j, \quad \sigma_{PD}^2(t_j) = C_j^2 \left[ \exp \left( b^2 \sum_{i=1}^j \sigma_i^2 \right) - 1 \right],$$

(21)
where the derivations of $\mu_{PD}(t_j)$ and $\sigma_{PD}^2(t_j)$ can be found in Appendix A. For each time node, if we consider the correlation coefficient of $V_j$ and and the probability of default at $t_j$ directly, namely $\rho(t_j)$ in the presence of parameter $b$, we have

$$\rho(t_j) = \frac{b\sqrt{\sum_{i=1}^{j} \sigma_i^2}}{\sqrt{\exp(b^2\sum_{i=1}^{j} \sigma_i^2) - 1}}. \quad (22)$$

The derivation of $\rho(t_j)$ is also given in Appendix A.

By virtue of (20) and (21), we can get the robust correlation coefficient $\bar{\rho}$ given by (15) and the profile multiplier $C_p$ given by (16) as follows

$$\bar{\rho} = \frac{b \sum_{j=1}^{K} C_j \sum_{i=1}^{j} \sigma_i^2}{\sum_{j=1}^{K} \left[ C_j \sqrt{\sum_{i=1}^{j} \sigma_i^2} \sqrt{\exp(b^2\sum_{i=1}^{j} \sigma_i^2) - 1} \right]} \quad (23)$$

$$C_p = \frac{\sum_{j=1}^{K} \left[ C_j \sqrt{\sum_{i=1}^{j} \sigma_i^2} \sqrt{\exp(b^2\sum_{i=1}^{j} \sigma_i^2) - 1} \right]}{\sum_{j=1}^{K} \left[ C_j \left( \sum_{i=1}^{j} \mu_i + V_0 \right) \right]} \quad (24)$$

Then, from (17), we can get the CVA ratio:

$$CVA_{ratio} = 1 + \bar{\rho} C_p = 1 + \frac{b \sum_{j=1}^{K} (C_j \sum_{i=1}^{j} \sigma_i^2)}{\sum_{j=1}^{K} \left[ C_j \left( \sum_{i=1}^{j} \mu_i + V_0 \right) \right]} \quad (25)$$

From (24), it is easy to see that the profile multiplier $C_p$ increases as $b$ increases in magnitude. Next we consider the sensitivity of the robust correlation coefficient $\bar{\rho}$ to the parameter $b$.

First we claim the magnitude of $\rho(t_j)$ decreases as $b$ increases in magnitude. This is sufficient to show when $b > 0$ because $\rho(t_j)$ is symmetric with respect to the origin. Let

$$x = b \sqrt{\sum_{j=1}^{K} \sigma_j^2}. \quad (26)$$

The square of equation (22) becomes

$$\rho^2(t_j) = \frac{x^2}{\exp(x^2) - 1}. \quad (27)$$

Taking the first order derivative of the above equation with respect to $x$, we can get

$$\frac{d(\rho^2(t_j))}{dx} = -\frac{2x \left[ (x^2 - 1) \exp(x^2) + 1 \right]}{[\exp(x^2) - 1]^2}.$$
It is easy to see that the numerator $2x [(x^2 - 1) \exp(x^2) + 1]$ is an increasing function and its value is 0 at $x = 0$. Therefore, the first order derivative of $\rho^2(t_j)$ to $x$ is less than 0 when $x > 0$. Clearly $\rho^2(t_j)$ is a decreasing function in $x$. Since $x$ is an increasing function in $b$, $\rho^2(t_j)$ is therefore a decreasing function in $b$. Moreover, when $x$ is positive, $\rho(t_j)$ is positive and has the same monotonicity as $\rho^2(t_j)$. Thus $\rho(t_j)$ decreases in magnitude as $b$ increases in magnitude.

Because $\bar{\rho}$ is a weighted average of $\rho(t_j)$ with all positive weights from (15), we can further claim that, for given $\sigma_V(t_j)$ and $\sigma_{PD}(t_j)$, the magnitude of the robust correlation coefficient $\bar{\rho}$ decreases as $b$ increases in magnitude.

One thing we want to point out is that based on (26) and (27), we can get

$$\lim_{b \to 0^-} \rho(t_i) = 1, \quad \lim_{b \to 0^+} \rho(t_i) = -1.$$ 

Therefore, we can get that

$$\lim_{b \to 0^+} \bar{\rho} = 1, \quad \lim_{b \to 0^-} \bar{\rho} = -1.$$ 

Therefore, the limit does not exist as $b$ approaches 0, and the Hull-White parametric approach could be very sensitive as $b$ approaches 0. We will give more discussion in the next section.

Further, we want to investigate the sensitivity of CVA to the underlying credit exposure volatility, i.e., CVA Vega. The CVA Vega is the partial derivative of CVA price with respect to the underlying volatility. Denote the underlying volatility as $\sigma_u$. The volatility of the credit exposure at time $t_i$, $\sigma_i$, is a function of $\sigma_u$, i.e. $\sigma_i = f_i(\sigma_u)$. Obviously $f_i(\sigma_u)$ is always non-negative and non-decreasing for all $t_i \leq T$. By virtue of (25), the CVA ratio is

$$CVA_{ratio} = 1 + \frac{b \sum_{j=1}^{K} (C_j \sum_{i=1}^{j} \sigma_i^2)}{\sum_{j=1}^{K} [C_j (\sum_{i=1}^{j} \mu_i + V_0)]} = 1 + \frac{b \sum_{j=1}^{K} (C_j \sum_{i=1}^{j} f_i(\sigma_u)^2)}{\sum_{j=1}^{K} [C_j (\sum_{i=1}^{j} \mu_i + V_0)]}.$$ (28)

Taking the derivative of equation (28) with respect to $\sigma_u$, we have

$$\frac{dCVA_{ratio}}{d\sigma_u} = \frac{b \sum_{j=1}^{K} (C_j \sum_{i=1}^{j} 2f_i(\sigma_u)f'_i(\sigma_u))}{\sum_{j=1}^{K} [C_j (\sum_{i=1}^{j} \mu_i + V_0)]}.$$ (29)

So the first order derivative or CVA Vega is positive when $b$ is greater than 0 and negative when $b$ is less than 0. In other words, $CVA_{ratio}$ is increasing with respect to underlying volatility when there is WWR and decreasing when RWR is in place.
6 Implications to CVA Estimation

From the analytical analysis in the last section, we can see that the single parameter $b$ in the Hull-White Approach actually plays two roles. On one hand, it models the same directional effect as the profile multiplier $C_p$, while on the other hand, it has negative impact on the robust correlation coefficient $\bar{\rho}$. Actually, from equations (24) and (23), we can see that the profile multiplier $C_p$ increases with the speed of $\sqrt{\exp(b^2 \sum_{i=1}^{L} \sigma_i^2)} - 1$ while the robust correlation coefficient $\bar{\rho}$ decreases with a slower speed of $b/\sqrt{\exp(b^2 \sum_{i=1}^{L} \sigma_i^2)} - 1$. Therefore, overall the increase of the profile multiplier $C_p$ dominates the decrease of the robust correlation coefficient $\bar{\rho}$ and the combined effect is positive with respect to $b$ (see (25)). In other words, the parameter $b$ has an overall positive impact on the DWR adjustment on the CVA. The numerical results to be presented in the next section (with the recovery rate $R = 0.4$) also verify the conclusion above (see Figure 2b and Figure 4b).

In the Hull-White Approach, the credit exposure $V(t)$ will need to be estimated first, and then the default probability hazard rate $h(t)$ will be estimated by calibrating with the market data. If the credit exposure $V(t)$ can be simulated or estimated easily, then the Hull-White Approach will be relatively easy to implement. In Hull-White (2012), the hazard rate is given in a very specific exponential form (see (3)) to describe the dependence of the counterparty default and the credit exposure. Other function forms to describe the dependence can be considered, too. For example, Ruiz, Boca and Pachón (2015) considers the probability of default as a function of some market factor and the authors examine function forms like power, exponential, logarithm and linear. In the Hull-White Approach, the dependence is modeled by one single parameter $b$, but the sensitivity with respect to $b$ seems not very clear. In addition, once the dependence structure changes, the calibration needs to be performed again to estimate the parameter $b$. Therefore, the Hull-White Approach should be used with caution.

Unlike the Hull-White Approach, the PCL Approach does not depend on any specific function form of the default hazard function. Moreover, in the PCL Approach, the profile multiplier $C_p$ and the robust correlation coefficient $\bar{\rho}$ are considered separately so that one can get more granular insight to the DWR in the CVA calculation. The profile multiplier $C_p$ can well capture the CVA effect caused by the volatilities of the exposure and the counterparty credit quality. The DWR due to the correlation between the credit exposure $V(t)$ and the default probability $q(t)$ is reflected in the robust correlation coefficient $\bar{\rho}$. To implement the approach, one need to generate or simulate both the credit exposure and the default probability at the same time. Based on these data, one can get the profile multiplier $C_p$, which does not change with the dependence between the credit exposure and the default probability. In other words,
$C_p$ stays the same across all levels of underlying correlations. The dependence will be characterized only by the robust correlation coefficient $\hat{\rho}$, which can be estimated with the efficient curve fitting algorithm given in Pang, Chen and Li (2015). When the risk manager does not have an accurate estimate of the correlation and only a confident interval is given on the underlying or the robust correlation coefficient, a corresponding confidence interval of CVA can be derived. Moreover, if the CVA value has to be re-estimated over time then this approach can be quite efficient. In particular, when the dependence structure changes due to market condition change, we simply change the value of the robust correlation coefficient $\hat{\rho}$ and do not need to re-simulate the credit exposure or the default probabilities.

The CVA DWR is only modeled by the single parameter $b$ in the Hull-White Approach. However, even when the $b$ value is fixed, the CVA price can decrease as the underlying volatility increases in the presence of RWR, and it can increase as the underlying volatility increases in the presence of WWR. For example, those phenomena are observed in the empirical results presented in Ruiz, Boca and Pachón (2015) (See Figure 9 in their paper). On the other hand, the PCL Approach with CVA DWR decomposition offers valuable details and provides intuitive explanation for those phenomena. From equation (29), we can see that, for CVA Vega, the change of the underlying volatility has a negative effect on the CVA price if there is a RWR ($b < 0$) and a positive effect on the CVA price if the WWR is presented ($b > 0$). In other words, from (29), it is clear that the CVA price not only depends on the value of $b$, but also depends on the portfolio profile (such as the underlying volatility). Therefore, the same value of $b$ in the Hull-White Approach does not always imply the same level of the DWR, and the portfolio profile does matter. Therefore, both the robust correlation and the profile multiplier can play key roles in DWR modeling. Considering only one of them may not be adequate.

In the Hull-White Approach, the dependence between the default and the exposure is modeled with functions in certain forms and there is no assumption to the distributions of the credit exposure or the default probability, so it is robust for all distributions. On the other hand, in the PCL Approach, the dependence is described by a robust correlation coefficient and it does not require any particular distribution assumption, either. The approach works as long as the first and second moments exist and the linear correlation is adequate to describe the underlying dependency. Actually, most elliptic distributions can meet those assumptions. Although the analysis in Section 5 is based on the assumptions of normal distributions and the recovery rate $R = 0$, the numerical results with $R \neq 0$ and no distribution assumption also indicate the same behavior. The numerical results and further discussions are presented in the next section.
7 Numerical Examples

Typically risk neutral parameter calibration is done based on market prices. It is a straightforward exercise to calibrate the means and standard deviations from the market price of the underlying exposure and credit spread of the counterparty as long as they are available. However the calibration of the robust correlation coefficient $\rho$ could be difficult because of the lack of market price of DWR CVA. One approach is to use the observed historical correlation $\rho$. In the Hull-White (2015), $a(t)$ is calibrated with the market observed credit spread of the counterparty. The authors also present two approaches to estimated the value of $b$, one is to use historical data of credit exposure and the credit spread, and the other involves subjective judgement of the DWR. In Ruiz, Boca and Pachón (2015), the authors presented some calibration results for models similar to the Hull-White Approach. Aiming to investigate the dependence of the default density to market factors, they proposed several function forms including linear, exponential and logarithm. The models are calibrated using market observed credit spread and different market factors such as equity price, foreign exchange and commodities. However, no calibrations was given based on the CVA prices.

In this paper, we focus on the analytical connection between the correlation approach and the parametric approach. Model calibration is left to a separate research. Instead, we illustrate the findings through a series of numerical examples for vanilla interest rate swaps. The numerical examples presented here are not based on normal assumptions, but we do observe phenomena that are consistent with the analytical results in Section 5.

7.1 Simulation Models

For illustration purposes we consider two very common risk free interest rate models. In particular, we use the short rate model proposed by Cox, Ingersoll and Ross (1985)(CIR)

$$dr_t = \kappa_r (\theta_r - r_t)dt + \sigma_r \sqrt{r_t}dW_t,$$  \hspace{1cm} (30)

and the model proposed by Vasicek (1977)

$$dr_t = \kappa_r (\theta_r - r_t)dt + \sigma_r dW_t,$$  \hspace{1cm} (31)

where $W_t$ is a standard Brownian motion, $\kappa_r$, $\theta_r$ and $\sigma_r$ the corresponding mean-reverting speed, mean-reverting level and volatility respectively.

Assume the trader has the exposures to the pay leg of 3-year fixed interest rate swaps with quarterly payments. The parameters are chosen to be $\kappa_r = 0.1$, $\theta_r = 0.05$ and $\sigma_r = 0.06$ and we assume that at time 0 the interest rate term structure is flat.
and equal to $\theta_r$. The fixed rate is also set to be $\theta_r$. The simulated future exposures have time dependent mean $\mu_V(t_i)$ and volatility $\sigma_V(t_i)$. This set of parameters is also used in Skoglund, Vestal and Chen (2013).

DWR dependency is modeled with the Hull-White Approach. Parameter $b$ ranges from $-0.4$ to $0.4$. A recovery rate of $R = 0.4$ is used. Based on the simulated data, for each given $b$ value, we calculate the the profile multiplier $C_p$, the robust correlation coefficient $\bar{\rho}$ and the combined CVA ratio $(1 + \bar{\rho}C_p)$. We investigate the sensitivities of them with respect to the parameter $b$ for both the CIR model and the Vasicek model. The numerical results are showed in Figures 1 to 4.

### 7.2 Numerical Results and Further Discussions

From Figure 1 and Figure 3, one can see that the profile multiplier $C_p$ increases as the magnitude of $b$ increases. This is true for both RWR and WWR. This result also holds for interest rates given by either the CIR model or the Vasicek model. In the presence of RWR, the magnitude of the profile multiplier ranges from 0 to slightly over 1. On the other hand, if WWR is presented, this number can reach almost 50 for the CIR model and 100 for the Vasicek model (see Figures 1 and 3). In other words, it seems like the profile multiplier $C_p$ is much more sensitive with respect to the parameter $b$ with WWR ($b > 0$) than that with RWR ($b < 0$).

The significant difference between RWR and WWR on the impact of the profile multiplier for the Hull-White Approach is very interesting. It means that the Hull-White Approach can catch the WWR well with a single variable of $b$. Since the RWR is not a big concern of the dealer or the regulators, so the significant lower sensitivity of the profile multiplier to the value of $b$ is acceptable. This behavior is mainly due to that the credit exposure truncation ($V^+(t) = \max\{V(t), 0\}$) will cause the volatility change in the default probabilities. When the WWR is presented ($b > 0$), the default probability volatility $\sigma_{PD}$ tends to be larger than that when a RWR is presented ($b < 0$). Now, from the profile multiplier definition (see (16)), it is easy to see that $C_P$ tends to be bigger when $b > 0$ and smaller when $b < 0$.

From Figure 2a and Figure 4a, we can see that the absolute value of the robust correlation coefficient decreases as $b$ increases in magnitude. This is consistent with the analysis in Section 5. Even though the exposures of vanilla swaps may not strictly follow our assumptions (4), (8) and (9) in Section 5, we observe the same monotonicity in this numerical study.

Further, from Figure 2b and Figure 4b, we can see that, the combined effect of CVA ratio is increasing in $b$. Obviously a WWR trade faces bigger potential credit loss if the counterparty default is more uncertain. Though the regulation does not award
RWR, risk managers can still benefit from RWR.

Another important finding is that, for the Hull-White Approach, when $b$ is positive but very close to 0, the CVA ratio tends to be more sensitive to $b$ (see Figure 2b and Figure 4b). From Figure 2b, we can see that for the CIR interest rate model, the CVA ratio changes from 1 to 5 when $b$ changes from 0 to 0.1. From Figure 4b, we can see that for the Vasicek interest rate model, the CVA ratio changes from 1 to 6.5 when $b$ changes from 0 to 0.1. As a reference, in Basel III (2011), the suggested CVA ratio is only 1.4 for the WWR. Therefore when the Hull-White Approach is used, the CVA with WWR is very sensitive to $b$ when $b$ is small. Therefore, more caution is needed for small values of $b$. A bad estimate of $b$ around 0 can be misleading.

8 Conclusion

In this paper, we developed a connection between the Hull-White parametric approach and the PCL correlation approach for CVA calculation. Hull and White (2012) provides a concise approach to capture DWR through a single parameter $b$. This paper provides an analytical anatomy of the dependence parameter $b$ in the Hull-White Approach through the effect of the robust correlation and the profile multiplier introduced by Pang, Chen and Li (2015) which shed more light on the parameter in terms of economic interpretation and effect from the dependency and underlying risks. We find that the dependence parameter $b$ in the Hull-White approach is not only affected by the robust correlation (an indicator of the underlying dependency), but also by the profile multiplier (severity of the underlying risk). It also shows that these two underlying effects are partially offset with each other when they are collectively reflected through the single parameter $b$. Whether this offset is desired or not requires further attention in model validation. In addition, the Hull-While model could be very sensitive while $b$ is positive and very close to 0. Our results help better understand the Hull-White Approach. In addition, in most cases, the PCL correlation approach by Pang, Chen and Li (2015) is easier to understand while also offers straightforward implementation.
Figure 1: Profile Multiplier - CIR Model

(a) RWR Profile Multiplier - CIR Model

(b) WWR Profile Multiplier - CIR Model
Figure 2: \( \hat{\rho} \) and CVA Ratio - CIR Model
Figure 3: Profile Multiplier - Vasicek Model

(a) RWR Profile Multiplier - Vasicek Model

(b) WWR Profile Multiplier - Vasicek Model
Figure 4: $\bar{\rho}$ and CVA Ratio - Vasicek Model
References


A The Derivation of $\sigma_{PD}^2(t_j)$ and $\rho(t_j)$

First, by virtue of (9), when $R = 0$, we can get

$$\mathbb{E} \left[ \exp \left( - \sum_{i=1}^{j} h_i \Delta t \right) \right] = e^{-s_j t_j}. \quad (32)$$

We approximate the left-hand side of the above equation with

$$\mathbb{E} \left[ \exp \left( - \sum_{i=1}^{j} h_i \Delta t \right) \right] \approx 1 - \mathbb{E} \left[ \Delta t \sum_{i=1}^{j} h_i \right]. \quad (33)$$

We want to point out that the approximation is good for the real market data. The error estimation for this approximation will be given in Appendix B.

Now, by virtue of (32) and (33), the market implied probability of default between time $t_{j-1}$ and $t_j$, denoted as $C_j$, can be estimated as the following

$$C_j \equiv e^{-s_{j-1} t_{j-1}} - e^{-s_j t_j}$$

$$= \mathbb{E} \left[ \exp(- \sum_{i=1}^{j-1} h_i \Delta t) \right] - \mathbb{E} \left[ \exp(- \sum_{i=1}^{j} h_i \Delta t) \right]$$

$$\approx \mathbb{E} \left[ \Delta t \sum_{i=1}^{j} h_i \right] - \mathbb{E} \left[ \Delta t \sum_{i=1}^{j-1} h_i \right]$$

$$= \mathbb{E} \left[ h_j \Delta t \right]. \quad (34)$$

Therefore, the probability of default at $t_j$ is given by $PD(t_j) = h_j \Delta t$. From the equation (34), we can get that

$$C_j = \mathbb{E} \left[ h_j \Delta t \right] = \mathbb{E} \left[ e^{a_j} \exp \left( b(V_0 + \sum_{i=1}^{j} X_i) \right) \Delta t \right]$$

$$= e^{a_j + b V_0} \exp \left( b \sum_{i=1}^{j} \mu_i + \frac{b^2}{2} \sum_{i=1}^{j} \sigma_i^2 \right) \Delta t. \quad (35)$$

Thus we have

$$e^{a_j} \Delta t = C_j e^{-b V_0} \exp \left( -b \sum_{i=1}^{j} \mu_i - \frac{b^2}{2} \sum_{i=1}^{j} \sigma_i^2 \right). \quad (36)$$

Now let us derive the second moment of $h_j \Delta t$.

$$\mathbb{E} \left[ (h_j \Delta t)^2 \right] = (\Delta t)^2 \mathbb{E} \left[ e^{2a_j} \exp \left( 2b(V_0 + \sum_{i=1}^{j} X_i) \right) \right]$$

$$= e^{2a_j} (\Delta t)^2 e^{2b V_0} \exp \left( 2b \sum_{i=1}^{j} \mu_i + 2b^2 \sum_{i=1}^{j} \sigma_i^2 \right). \quad (37)$$
Plug equation (36) into equation (37) and we can get

\[ E \left[ (h_j \Delta t)^2 \right] = C_j^2 \exp \left( b^2 \sum_{i=1}^{j} \sigma_i^2 \right). \]  

(38)

Hence

\[ \sigma_{PD}^2(t_j) = E \left[ (h_j \Delta t)^2 \right] - (E[h_j \Delta t])^2 = C_j^2 \left[ \exp \left( b^2 \sum_{i=1}^{j} \sigma_i^2 \right) - 1 \right]. \]

Next we derive the formula for \( \rho(t_j) \). We need the following lemma:

**Lemma 1** (Stein’s Lemma). Assume that a random variable \( X \) follows a normal distribution \( N(\mu, \sigma^2) \). Then, for a function \( g \) such that both \( E[g(X)(X - \mu)] \) and \( E[g'(X)] \) exist, we have

\[ E[g(X)X] = \sigma^2 E[g'(X)] + \mu E[g(X)]. \]  

(39)

Proof of Stein’s Lemma can be found in Stein (1981).

By virtue of the Stein’s Lemma, we can get the covariance of the credit exposure \( V_j \) and the probability of default \( PD(t_j) = h_j \Delta t \) at the time \( t_j \) as follows

\[
\text{Cov}(V_j, PD(t_j)) = \Delta t \text{Cov}(V_j, e^{a_j + bV_j})
\]

\[
= \Delta t \left( E[e^{a_j + bV_j}V_j] - E[e^{a_j + bV_j}]E[V_j] \right)
\]

\[
= \Delta t \sigma^2_{V_j} E[b e^{a_j + bV_j}]
\]

\[
= \sigma^2_{PD} \sum_{i=1}^{j} \sigma_i^2 E[h_j \Delta t] = C_j b \sum_{i=1}^{j} \sigma_i^2.
\]

Hence

\[
\rho(t_j) = \frac{\text{Cov}(V_j, PD(t_j))}{\sigma_{V_j} \sigma_{PD}(t_j)}
\]

\[
= \frac{C_j b \sum_{i=1}^{j} \sigma_i^2}{C_j \sqrt{\exp(b^2 \sum_{i=1}^{j} \sigma_i^2)} - 1 \sqrt{\sum_{i=1}^{j} \sigma_i^2}}
\]

\[
= \frac{b \sqrt{\sum_{i=1}^{j} \sigma_i^2}}{\sqrt{\exp(b^2 \sum_{i=1}^{j} \sigma_i^2)} - 1}.
\]
B Error of Approximation (33)

In Appendix A, we use the following approximation (equation (33)):

\[ E \left[ \exp \left( - \sum_{i=1}^{j} h_i \Delta t \right) \right] = e^{-s_f t_f}. \]

Here we will investigate the accuracy about the above approximation with some numerical results. In particular, we set the initial value of the portfolio as $10,000. On each day there is a profit or loss that follows a normal distribution with 0 mean and $100 standard deviation. The time horizon we use is one year or 252 trading days. We choose \( b \) to be 0.4. 100,000 paths are used in this simulation. The following figures show the numerical results for CDS spread up to 300 basis points.

![Max Approximation Error](image)

**Figure 5:** Max Error

As we can tell from Figure 5, this approximation error increases as CDS spread increases but the maximum error is only about $5.2 \times 10^{-4}$, which is very very small. On the other hand, Figure 6 (source www.bondsonline.com, last accessed Feb. 2016) is an example of market quoted CDS spread, which shows for one year time horizon, 300 basis points can cover all rating grades except Caa/CCC. Therefore, that the first order approximation (33) we use in Section 5 can be safely assumed for most of the investment grade products.
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Figure 6: Example of Market Quoted CDS Spread