Viscosity Solution of Optimal Stopping Problem for Stochastic Systems with Bounded Memory

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Abstract

We consider a finite time horizon optimal stopping problem for a system of stochastic functional differential equations with a bounded memory. Under some sufficiently smooth conditions, a Hamilton-Jacobi-Bellman (HJB) variational inequality for the value function is derived via dynamical programming principle. It is shown that the value function is the unique viscosity solution of the HJB variational inequality.

KEYWORDS: optimal stopping, stochastic control, stochastic functional differential equations.

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1 Introduction

Optimal control and optimal stopping problems over a finite or an infinite time horizon for Itô’s diffusion processes arise in many areas of science, engineering, and finance (see e.g. Fleming and Soner [14], Øksendal [35], Shiryaev [39], Karazas and Shreve [19] and references contained therein). The value function of these problems are normally expressed as a viscosity or a generalized solution of a Hamilton-Jacobi-Bellman equation (HJBE) or a Hamilton-Jacobi-Bellman variational inequality (HJBVI) that involves a second order parabolic or elliptic partial differential equation in a finite dimensional Euclidean space (see e.g. Lions [30] and [32]).

In an attempt to achieve better accuracy and to account for the delayed effect of the state variables in the modelling of real world stochastic control problems, the stochastic delay equations and controlled stochastic delay equations have been the subject of intensive studies in recent years by many researchers such as Elsanousi [11], Larrssen [25], Elsanousi et al [13], Øksendal and Sulem [37], Larrssen and Risebro [27], and Bauer and Rieder [5].

The controlled or uncontrolled stochastic delay equations considered by the aforementioned researchers are described by the following special classes of equations that contain discrete and averaged delays:

\[
\begin{align*}
\frac{dX}{ds} &= \alpha(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda \theta} X(s+\theta) d\theta, u(s)) ds \\
&+ \beta(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda \theta} X(s+\theta) d\theta, u(s)) dW(s), \quad s \in [t, T],
\end{align*}
\]

or

\[
\begin{align*}
\frac{dX}{ds} &= \alpha(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda \theta} X(s+\theta) d\theta) ds \\
&+ \beta(s, X(s), X(s-r), \int_{-r}^{0} e^{\lambda \theta} X(s+\theta) d\theta) dW(s), \quad s \in [t, T].
\end{align*}
\]

In the above equations, \(W(\cdot) = \{W(s), s \geq 0\}\) is an \(m\)-dimensional standard Brownian motion defined on a certain filtered probability space \((\Omega, \mathcal{F}, P, \mathcal{F})\), \(u(\cdot) = \{u(s), s \in [t, T]\}\) is a control process taking values in the control set \(U\) in an Euclidean space, \(\alpha\) and \(\beta\) are \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times m}\)-valued functions defined on

\[ [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times U, \]

or

\[ [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \]

and \(\lambda > 0\) is a given constant.

Based on the above two equations or their variants, Larssen [25] obtained an HJB equation for an optimal control problem, Elsanousi et al [13] considered a singular control problem and obtained certain explicitly available solutions, Øksendal and Sulem [37] derived the maximum principle for the optimal
stochastic control. If the dynamics of the control problem with delay exhibit a special structure, Larssen and Risebro [27] and Bauer and Rieder [5] showed that the value function actually lives in a finite-dimensional space and the original problem can be reduced to a classical stochastic control problem without delay. Elsanousi and Larssen [12] treated an optimal control problem and its applications to consumption for (1) under partial observation. We also mention that optimal stopping problems were studied in Elsanousi’s unpublished dissertation [11] for such special type of equations.

This paper extends the results obtained for finite dimensional diffusion processes and stochastic delay equations described in (2) and investigates an optimal stopping problem over a finite time horizon for a general system of stochastic functional differential equations (SFDE) described below:

\[ dX(s) = f(s, X_s)dt + g(s, X_s)dW(s), \quad s \in [t, T], \]

where \( T > 0 \) and \( t \in [0, T] \), respectively, denote the terminal time and an initial time of the optimal stopping problem. Again, \( W(\cdot) = \{W(s), s \geq 0\} \) is a standard \( m \)-dimensional Brownian motion, and the drift \( f(s, X_s) \) and the diffusion coefficient \( g(s, X_s) \) (taking values in \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times m} \), respectively) depend explicitly on the segment \( X_s \) of the state process \( X(\cdot) = \{X(s), s \in [t-r, T]\} \) over the time interval \([s-r, s]\), where \( X_s : [-r, 0] \to \mathbb{R}^n \) is defined by \( X_s(\theta) = X(s + \theta), \theta \in [-r, 0] \). The consideration of such a system enables us to model many real world problems that have aftereffects (see e.g. Kolmanovsky and Shaikhet [24] and references contained therein for application examples). It is clear that this equation also includes (2) as a special case and many other equations that can not be modelled in the form of (2). When \( r = 0 \), it is also clear that the SFDE (3) reduces to the following Itô’s diffusion process (without delay):

\[ dX(s) = f(s, X(s))ds + g(s, X(s))dW(s), \quad s \in [t, T]. \]

This paper treats a finite time horizon optimal stopping problem (see Section 2 for the problem statement). We derive an infinite dimensional HJB variational inequality (HJBVI) for the value function via a dynamic programming principle (see e.g. Larssen [25]). It is shown that the value function is the unique viscosity solution of the HJBVI. The proof of uniqueness involves embedding the function space \( W^{1,2}((-r, 0); \mathbb{R}^n) \) into the Banach space \( C([-r, 0]; \mathbb{R}^n) \) and extending the concept of viscosity solution for controlled Itô’s diffusion process developed by Crandall et al [9], and Lions [30] and [32] to an infinite dimensional setting.

Although infinite dimensional HJBVIs for optimal stopping problems and their applications to pricing of American option have been studied very recently by a few researchers, they either considered stochastic delay equation of special form (2)(see e.g. Gapeev and Reiss [16] and [17]) or stochastic equations in Hilbert spaces (see e.g. Gatarek and Świech [15] and Barbu and Marinelli [4]). This paper differs from the aforementioned papers in the following significant ways: i) The segmented solution process \( \{X_s, s \in [t, T]\} \) is a strong Markov process in the Banach space \( C([-r, 0]; \mathbb{R}^n) \) whose norm is not differentiable and is therefore more difficult to handle than any Hilbert space considered in
and [4]; ii) the infinite-dimensional HJBVI uniquely involves the extensions $D V(t, \psi)$ and $D^2 V(t, \psi)$ of first and second order Fréchet derivatives $D V(t, \psi)$ and $D^2 V(t, \psi)$ from $C^*$ and $C^1$ to $(C \oplus B)^*$ and $(C \oplus B)^1$ (see Subsection 3.1 for definitions of these spaces), respectively; and iii) the infinite-dimensional HJBVI also involves the infinitesimal generator $S V(t, \psi)$ of the semigroup of shift operators value functions that does not appear in the special class of equations (2) in the aforementioned papers.

This paper is organized as follows. The notation and preliminary results that are needed for formulating the optimal stopping problem as well as the problem statement are contained in Section 2. In Section 3, the HJBVI for the value function is heuristically derived using Bellman’s type dynamic programming principle. The verification theorem is also proved there. In Section 4, the continuity of the value function is proved. Although continuous, the value function is not known to be smooth enough to be a classical solution of the HJBVI in general cases. It is shown in Section 4, however, that the value function is the unique viscosity solution of the HJBVI.

2 The Optimal Stopping Problem

Let $r > 0$ be a fixed constant, and let $J = [-r, 0]$ denote the duration of the bounded memory of the stochastic functional differential equations considered in this paper. For the sake of simplicity, denote $C(J; \mathbb{R}^n)$, the space of continuous functions $\phi : J \to \mathbb{R}^n$, by $C$. Note that $C$ is a real separable Banach space under the sup-norm defined by

$$||\phi|| = \sup_{\theta \in J} |\phi(\theta)|, \quad \phi \in C$$

where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^n$.

In addition to the space $C$, we also consider $L^2(J; \mathbb{R}^n)$, the Hilbert space of Lebesgue squared-integrable functions $\phi : J \to \mathbb{R}^n$ equipped with the inner product $\langle \cdot, \cdot \rangle$ and the Hilbertian norm $\| \cdot \|_2$ defined by

$$\langle \phi | \psi \rangle = \int_{-r}^0 \langle \phi(\theta), \psi(\theta) \rangle d\theta \quad \text{and} \quad \| \phi \|_2 = (\langle \phi | \phi \rangle)^{\frac{1}{2}}, \quad \forall \phi, \psi \in L^2(J, \mathbb{R}^n),$$

where $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$.

Note that the space $C$ can be continuously embedded into $L^2(J; \mathbb{R}^n)$ (see e.g. Rudin [38]). In fact, it is easy to verify that

$$\| \phi \|_2 \leq \sqrt{r} \| \phi \|, \quad \forall \phi \in C.$$

Convention 2.1 Throughout the rest of the paper, let $T > 0$ denote the terminal time and $t \in [0, T]$ be an initial time of the optimal stopping problem. We shall use the following conventional notation for functional differential equations (see Hale [18]):

If $\psi \in C([t-r, T]; \mathbb{R}^n)$ and $s \in [t, T]$, let $\psi_s \in C$ be defined by

$$\psi_s(\theta) = \psi(s + \theta), \quad \forall \theta \in J.$$
In addition, throughout the rest of the paper, we will use $K$ and $\Lambda$ to denote generic constants and their values may change from line to line.

Let $\{W(s), s \geq 0\}$ be an $m$-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F})$, where $\mathcal{F}$ denotes the smallest $\sigma$-field that contains all subsets of $\Omega$, and $\mathcal{F} = \{\mathcal{F}(s), s \geq 0\}$ is the $P$-augmentation of the natural filtration $\{\mathcal{F}^W(s), s \geq 0\}$ generated by the Brownian motion $\{W(s), s \geq 0\}$, i.e., if $s \geq 0$,

$$\mathcal{F}^W(s) = \sigma\{W(t), 0 \leq t \leq s\}$$

and

$$\mathcal{F}(s) = \mathcal{F}^W(s) \lor \{A \subset \Omega|\exists B \in \mathcal{F} \text{ such that } A \subset B \text{ and } P(B) = 0 \},$$

where the operator $\lor$ denotes that $\mathcal{F}(s)$ is the smallest $\sigma$-algebra such that $\mathcal{F}^W(s) \subset \mathcal{F}(s)$ and

$$\{A \subset \Omega|\exists B \in \mathcal{F} \text{ such that } A \subset B \text{ and } P(B) = 0 \} \subset \mathcal{F}(s).$$

Consider the following system of stochastic functional differential equations:

$$dX(s) = f(s, X_s)ds + g(s, X_s)dW(s), \ s \in [t, T]; \quad (4)$$

with the initial function $X_t = \psi_t$, where $\psi_t$ is a given $C$-valued random variable that is $\mathcal{F}(t)$-measurable. Here, $f : [0, T] \times C \to \mathbb{R}^n$ and $g : [0, T] \times C \to \mathbb{R}^{n \times m}$ are given deterministic functionals.

Let $L^2(\Omega, C)$ be the space of $C$-valued random variables $\Xi : \Omega \to C$ such that

$$||\Xi||_{L^2} = \left(\int_{\Omega} ||\Xi(\omega)||^2 dP(\omega)\right)^{\frac{1}{2}} < \infty.$$

Let $L^2(\Omega, C; \mathcal{F}(t))$ be the space of those $\Xi \in L^2(\Omega, C)$ which are $\mathcal{F}(t)$-measurable.

**Definition 2.2** A process $\{X(s; \cdot, \psi_t), s \in [t - r, T]\}$ is said to be a (strong) solution of (4) on the interval $[t - r, T]$ and through the initial datum $(t, \psi_t) \in [0, T] \times L^2(\Omega, C; \mathcal{F}(t))$ if it satisfies the following conditions:

1. $X_t(\cdot; t, \psi_t) = \psi_t$;

2. $X(s; t, \psi_t)$ is $\mathcal{F}(s)$-measurable for each $s \in [t, T]$;

3. The process $\{X(s; \cdot, \psi_t), s \in [t, T]\}$ is continuous and it satisfies the following stochastic integral equation $P$-a.s.

$$X(s) = \psi_t(0) + \int_t^s f(\lambda, X_\lambda)d\lambda + \int_t^s g(\lambda, X_\lambda)dW(\lambda), \ s \in [t, T]. \quad (5)$$
In addition, the (strong) solution process \( \{X(s; t, \psi_t), s \in [t-r, T]\} \) of (4) is said to be (strongly) unique if \( \{\tilde{X}(s; t, \psi_t), s \in [t-r, T]\} \) is also a (strong) solution of (4) on \([t-r, T]\) and through the same initial datum \((t, \psi_t)\), then
\[
P\{X(s; t, \psi_t) = \tilde{X}(s; t, \psi_t), \forall s \in [t, T]\} = 1.
\]
Throughout the rest of the paper, we assume that the functionals \( f : [0, T] \times C \to \mathbb{R}^n \), and \( g : [0, T] \times C \to \mathbb{R}^{n \times m} \) are continuous functionals and they satisfy the following conditions (Assumption 2.3 & 2.4) to ensure the existence and uniqueness of a (strong) solution \( \{X(s; t, \psi_t), s \in [t-r, T]\} \) for each \((t, \psi_t) \in [0, T] \times L^2(\Omega, C; \mathcal{F}(t))\). (See Mohammed [33, 34].)

**Assumption 2.3** There exists a constant \( K > 0 \) such that
\[
|f(t, \varphi) - f(s, \phi)| + |g(t, \varphi) - g(s, \phi)| \\
\leq K([|t-s| + \|\varphi - \phi\|]) \quad \forall (t, \varphi), (s, \phi) \in [0, T] \times C.
\]

**Assumption 2.4** There exists a constant \( K > 0 \) such that
\[
|f(t, \phi)| + |g(t, \phi)| \leq K(1 + \|\phi\|) \quad \forall (t, \phi) \in [0, T] \times C.
\]

Let \( \{X(s; t, \psi_t), s \in [t, T]\} \) be the solution of (4) through the initial datum \((t, \psi_t) \in [0, T] \times C\). We consider the corresponding \( C \)-valued process \( \{X_s(t, \psi_t), s \in [t, T]\} \) defined by
\[
X_s(t, \psi_t) \equiv X(s + \theta; t, \psi_t), \quad \forall \theta \in \mathbb{J}.
\]
(6)

For each \( t \in [0, T] \), define \( G(t) \equiv \bigcup_{s \in [t, T]} G(t, s) \) where \( G(t, s) \) is defined by
\[
G(t, s) = \sigma(X(u; t, \psi_t), t \leq u \leq s).
\]
Note that, it can be shown that for each \( s \in [t, T], \)
\[
G(t, s) = \sigma(X_u(t, \psi_t), t \leq u \leq s).
\]
This is due to the sample path’s continuity of the process \( \{X(s; t, \psi_t), s \in [t, T]\} \).

One can then establish the following Markov property (see Mohammed [33], [34]).

**Theorem 2.5** Let Assumptions 2.3 and 2.4 hold. Then the corresponding \( C \)-valued solution process of (4) describes a \( C \)-valued Markov process in the following sense:

For any \((t, \psi_t) \in [0, T] \times L^2(\Omega, C)\), the Markovian property
\[
P\{X_s(t, \psi_t) \in B|G(t, \alpha)\} = P\{X_s(t, \psi_t) \in B|X_{\alpha}(t, \psi_t)\} \equiv p(\alpha, X_{\alpha}(t, \psi_t), s, B)
\]
holds a.s. for \( t \leq \alpha \leq s \) and \( B \in \mathcal{B}(C) \), where \( \mathcal{B}(C) \) is the Borel \( \sigma \)-algebra of subsets of \( C \).
In the above, the function \( p : [t,T] \times \mathbb{C} \times [t,T] \times \mathcal{B}(\mathbb{C}) \to [0,1] \) denotes the transition probabilities of the \( \mathbb{C} \)-valued Markov process \( \{ X_s(t, \psi_t), s \in [t,T]\} \).

A random function \( \tau : \Omega \to [0,\infty] \) is said to be a \( \mathcal{G}(t) \)-stopping time if

\[
\{ \tau \leq s \} \in \mathcal{G}(t, s), \quad \forall s \geq t.
\]

Let \( \mathcal{T} \) be the collection of all \( \mathcal{G}(t) \)-stopping times, and let \( \mathcal{T}_t^T \) be the collection of all \( \mathcal{G}(t) \)-stopping times \( \tau \in \mathcal{T} \) such that \( t \leq \tau \leq T \) a.s.. For each \( \tau \in \mathcal{T}_t^T \), let the sub-\( \sigma \)-algebra \( \mathcal{G}(t, \tau) \) of \( \mathcal{F} \) be defined by

\[
\mathcal{G}(t, \tau) = \{ A \in \mathcal{F} | A \cap \{ t \leq \tau \leq s \} \in \mathcal{G}(t, s) \ \forall s \in [t,T] \}.
\]

With a little bit more effort, one can show that the corresponding \( \mathbb{C} \)-valued solution process of (4) is also a strong Markov process in \( \mathbb{C} \). That is

\[
P\{ X_s(t, \psi_t) \in B | \mathcal{G}(t, \tau) \} = P\{ X_s(t, \psi_t) \in B | X_\tau(t, \psi_t) \} \equiv p(\tau, X_\tau(t, \psi_t), s, B)
\]

holds a.s. for all \( \tau \in \mathcal{T}_t^T \), all deterministic \( s \in [\tau, T] \), and \( B \in \mathcal{B}(\mathbb{C}) \).

If the drift \( f \) and the diffusion coefficient \( g \) are time-independent, i.e.,

\[
f(s, \phi) \equiv f(\phi) \quad \text{and} \quad g(s, \phi) \equiv g(\phi),
\]

then (4) reduces to the following autonomous system:

\[
dX(s) = f(X_s)ds + g(X_s)dW(s). \tag{7}
\]

In this case, we usually assume the initial datum \((t, \psi_t) = (0, \psi)\) and denote the solution process of (7) through \((0, \psi)\) and on the interval \([-r,T]\) by \( \{ X(s; \psi), s \in [-r,T] \} \). Then the corresponding \( \mathbb{C} \)-valued solution process \( \{ X_s(\psi), s \in [-r,T] \} \) of (7) is a strong Markov process with time-homogeneous probability transition function \( p(\psi, s, B) \equiv p(0, \psi, s, B) = p(t, \psi, t+s, B) \) for all \( s, t \geq 0, \psi \in \mathbb{C} \), and \( B \in \mathcal{B}(\mathbb{C}) \).

Assume \( L \) and \( \Psi \) are two \( \| \cdot \|_2 \)-Lipschitz continuous real-valued functionals on \([0,T] \times \mathbb{C}\) with at most polynomial growth in \( L^2(J; \mathbb{R}^n) \). In other words, there exist positive constants \( K, \Lambda, \) and \( k \geq 1 \) such that

\[
|L(t, \psi) - L(s, \phi)| + |\Psi(t, \psi) - \Psi(s, \phi)| \\
\leq K(|t - s| + \| \psi - \phi \|_2) \\
\leq K(|t - s| + \| \psi - \phi \|) \quad \forall (t, \psi), (s, \phi) \in [0,T] \times \mathbb{C}. \tag{8}
\]

and

\[
|L(t, \phi)| + |\Psi(t, \phi)| \leq \Lambda(1 + \| \phi \|_2)^k, \quad \forall (t, \phi) \in [0,T] \times \mathbb{C}. \tag{9}
\]

Given the initial datum \((t, \psi) \in [0,T] \times \mathbb{C}\), our objective is to find an optimal stopping time \( \tau^* \in \mathcal{T}_t^T \) that maximizes the following expected performance index:

\[
J(\tau; t, \psi) \equiv \mathbb{E} \left[ \int_\tau^T e^{-\rho(s-t)}L(s, X_s)ds + e^{-\rho(\tau-t)}\Psi(\tau, X_\tau) \right], \tag{10}
\]

where \( \rho > 0 \) denotes a discount factor. In this case, the value function \( V : [0,T] \times \mathbb{C} \to \mathbb{R} \) is defined to be

\[
V(t, \psi) \equiv \sup_{\tau \in \mathcal{T}_t^T} J(\tau; t, \psi). \tag{11}
\]
For the autonomous case, i.e.,
\[dX(s) = f(X_s)dt + \sigma(X_s)dW(s), \quad s \in [t, T],\]
the following optimal stopping problem is a special case of what will be treated in this paper: Find an optimal stopping time \(\tau^* \in T_0^T\) that maximizes the following expected performance index:
\[J(\tau; \psi) \equiv \mathbb{E} \left[ \int_0^\tau e^{-\rho s} L(X_s)ds + e^{-\rho \tau} \Psi(X_\tau) \right].\]
In this case, the value function \(V: \mathbb{C} \to \mathbb{R}\) is defined to be
\[V(\psi) \equiv \sup_{\tau \in T_0^T} J(\tau; \psi).\]

3 HJB Variational Inequality

3.1 The Infinitesimal Generator

Let \(C^*\) and \(C^\dagger\) be the space of bounded linear functionals \(\Phi: \mathbb{C} \to \mathbb{R}\) and bounded bilinear functionals \(\tilde{\Phi}: \mathbb{C} \times \mathbb{C} \to \mathbb{R}\), of the space \(\mathbb{C}\), respectively. They are equipped with the operator norms which will be, respectively, denoted by \(\| \cdot \|^*\) and \(\| \cdot \|^\dagger\).

Let \(B = \{v \mathbf{1}_{(0)}, v \in \mathbb{R}^n\}\), where \(\mathbf{1}_{(0)}: [-r, 0] \to \mathbb{R}\) is defined by
\[\mathbf{1}_{(0)}(\theta) = \begin{cases} 0 & \text{for } \theta \in (-r, 0), \\ 1 & \text{for } \theta = 0. \end{cases}\]

We form the direct sum
\[\mathbb{C} \oplus B = \{\phi + v \mathbf{1}_{(0)} \mid \phi \in \mathbb{C}, v \in \mathbb{R}^n\}\]
and equip it with the norm, also denoted by \(\| \cdot \|\), defined by
\[\|\phi + v \mathbf{1}_{(0)}\| = \sup_{\theta \in [-r, 0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbb{C}, v \in \mathbb{R}^n.\]

Again, let \((\mathbb{C} \oplus B)^*\) and \((\mathbb{C} \oplus B)^\dagger\) be spaces of bounded linear and bilinear functionals of \(\mathbb{C} \oplus B\), respectively.

The following two results can be found in Lemma 3.1 and Lemma 3.2 on pp 79-83 of Mohammed [33].

If \(\Gamma \in \mathbb{C}^*\), then \(\Gamma\) has a unique (continuous) bilinear extension \(\tilde{\Gamma}: (\mathbb{C} \oplus B)^* \to \mathbb{R}\) satisfying the following weak continuity property:

(W1) If \(\{\xi^{(k)}\}_{k=1}^\infty\) is a bounded sequence in \(\mathbb{C}\) such that \(\xi^{(k)}(\theta) \to \xi(\theta)\) as \(k \to \infty\) for all \(\theta \in \mathbb{I}\) for some \(\xi \in \mathbb{C} \oplus B\), then \(\Gamma(\xi^{(k)}) \to \tilde{\Gamma}(\xi)\) as \(k \to \infty\). The extension map \(\mathbb{C}^* \to (\mathbb{C} \oplus B)^*, \Gamma \mapsto \tilde{\Gamma}\) is a linear isometric injective map.
If $\Gamma \in \mathbf{C}^\dagger$, then $\Gamma$ has a unique (continuous) linear extension $\bar{\Gamma} \in (\mathbf{C} \oplus \mathbf{B})^\dagger$ satisfying the following weak continuity property:

(W2) If $\{\xi^{(k)}\}_{k=1}^\infty$ and $\{\zeta^{(k)}\}_{k=1}^\infty$ are bounded sequences in $\mathbf{C}$ such that $\xi^{(k)}(\theta) \to \xi(\theta)$ and $\zeta^{(k)}(\theta) \to \zeta(\theta)$ as $k \to \infty$ for all $\theta \in \mathcal{I}$ for some $\xi, \zeta \in \mathbf{C} \oplus \mathbf{B}$, then $\Gamma(\xi^{(k)}, \zeta^{(k)}) \to \bar{\Gamma}(\xi, \zeta)$ as $k \to \infty$.

For a sufficiently smooth functional $\Phi : \mathbf{C} \to \mathbf{R}$, we can define its Fréchet derivatives with respect to $\phi \in \mathbf{C}$. (For more details about Fréchet derivatives, please refer to [2].) From the results stated above, we know that its first order derivatives with respect to $\phi$ defined by $\Gamma(\phi)$ then

$$\frac{\partial \Gamma}{\partial \phi}(\phi) = \lim_{h \to 0^+} \frac{1}{h} \left[ \Phi(\phi_h) - \Phi(\phi) \right]$$

for all $\phi \in \mathbf{C}$, where $\phi_h : [-r, T] \to \mathbf{R}^n$ is an extension of $\phi : [-r, 0] \to \mathbf{R}^n$ defined by

$$\phi(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0), \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and again $\phi_t \in \mathbf{C}$ is defined by

$$\phi_t(\theta) = \phi(t + \theta), \quad \theta \in [-r, 0].$$

Let $\mathcal{D}(\mathcal{S})$, the domain of the operator $\mathcal{S}$, be the set of $\Phi : \mathbf{C} \to \mathbf{R}$ such that the above limit exists for each $\phi \in \mathbf{C}$. Define $\mathcal{D}(\mathcal{S})$ as the set of all functionals $\Psi : [0, T] \times \mathbf{C} \to \mathbf{R}$ such that $\Psi(t, \cdot) \in \mathcal{D}(\mathcal{S}), \forall t \in [0, T]$.

Throughout the end, let $C_{lip}^1([0, T] \times \mathbf{C})$ be the space of functions $\Phi : [0, T] \times \mathbf{C} \to \mathbf{C}$ which satisfies the following conditions:

1. $\frac{\partial \Phi}{\partial t} : [0, T] \times \mathbf{C} \to \mathbf{R}$, $D\Phi : [0, T] \times \mathbf{C} \to \mathbf{C}^\ast$ and $D^2\Phi : [0, T] \times \mathbf{C} \to \mathbf{C}^\dagger$ exist and are continuous.

2. Its second order Fréchet derivative $D^2\Phi$ satisfies the following global Lipschitz condition:

$$\|D^2\Phi(t, \phi) - D^2\Phi(t, \varphi)\| \leq K\|\phi - \varphi\| \quad \forall t \in [0, T], \phi, \varphi \in \mathbf{C}.$$ 

The following example shows that $\mathcal{D}(\mathcal{S})$ is not a subset of $C_{lip}^2(\mathbf{C})$. Therefore, it is not redundant to require that a function $\Phi \in C_{lip}^{1,2}([0, T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$ for deriving the infinite-dimensional HJB variational inequality.

**Example.** Assume $n = 1$. Let $\Phi : \mathbf{C} \to \mathbf{R}$ be independent of the time variable and be defined by

$$\Phi(\phi) = \phi \left( -\frac{r}{2} \right), \quad \phi \in \mathbf{C}.$$
Then it is clear that $\Phi \in C^2(\mathbb{C})$ but $\Phi \notin \hat{D}(\mathbb{S})$, since $\forall \varphi \in \mathbb{C}$,

$$D\Phi(\varphi)(\varphi) = \varphi \left(-\frac{r}{2}\right),$$

$$D^2\Phi(\varphi)(\varphi, \varphi) = 0,$$

and $D^2\Phi$ is globally Lipschitz. However,

$$S(\Phi)(\varphi) = \lim_{t \to 0^+} \frac{1}{t} \left[ \Phi(\tilde{\varphi}_t) - \Phi(\varphi) \right] = \lim_{t \to 0^+} \frac{\tilde{\varphi}(t - \frac{r}{2}) - \varphi(-\frac{r}{2})}{t},$$

which exists only if $\tilde{\varphi}$ has a right derivative at $-\frac{r}{2}$.

Let $\Phi : [0, T] \times \mathbb{C} \to \mathbb{R}$ be a Borel measurable function and consider the following two smoothness conditions:

**Smoothness Condition (i).** $\Phi \in C^{1,2}_{lip}([0, T] \times \mathbb{C})$.

**Smoothness Condition (ii).** $\Phi \in \hat{D}(\mathbb{S})$, i.e., $\Phi(t, \cdot) \in \hat{D}(\mathbb{S})$ for each $t \in [0, T]$.

The following result will be used later on in this paper.

**Theorem 3.1 (Mohammed [33], [34])** Suppose that $\Phi \in C([0, T] \times \mathbb{C})$ satisfies Smoothness conditions (i) and (ii). Let $\{X_s, s \in [t, T]\}$ be the $\mathbb{C}$-valued Markov solution process of Equation (4) with the initial data $(t, \varphi_t) \in [0, T] \times \mathbb{C}$.

Then

$$\lim_{\epsilon \to 0} \frac{E[\Phi(t + \epsilon, X_{t+\epsilon})] - \Phi(t, \varphi_t)}{\epsilon} = \frac{\partial}{\partial t} \Phi(t, \varphi_t) + S(\Phi)(t, \varphi_t) + D\Phi(t, \varphi_t)(f(t, \varphi_t)1_{\{0\}}) + \frac{1}{2} \sum_{j=1}^{m} D^2\Phi(t, \varphi_t)(g(t, \varphi_t)(e_j)1_{\{0\}}, g(t, \varphi_t)(e_j)1_{\{0\}}),$$

where $e_j, j = 1, 2, \cdots, m$, is the $j$th vector of the standard basis in $\mathbb{R}^m$.

**3.2 Heuristic Derivation**

We consider the following HJBVI:

$$\max \left\{ \Psi - V, \frac{\partial V}{\partial t} + AV + L - \rho V \right\} = 0$$

on $[0, T] \times \mathbb{C}$, where

$$AV(t, \psi) \equiv S(V)(t, \psi) + D\Phi(t, \psi)(f(t, \psi)1_{\{0\}}) + \frac{1}{2} \sum_{i=1}^{m} D^2\Phi(t, \psi)(g(t, \psi)(e_i)1_{\{0\}}, g(t, \psi)(e_i)1_{\{0\}}).$$
The above inequality shall be interpreted as follows.

\[
\frac{\partial V}{\partial t} + AV + L - \rho V \leq 0 \quad \text{and} \quad V \geq \Psi \quad (19)
\]

and

\[
\left( \frac{\partial V}{\partial t} + AV + L - \rho V \right) (V - \Psi) = 0 \quad (20)
\]
on \([0, T] \times \mathbb{C}\), where \(A\) is defined by (18).

We heuristically derive the above variational inequality as follows. A rigorous derivation will be provided as a byproduct when we prove that the value function is a viscosity solution of the HJBVI in the next section. First, we prove that \(V(t, \psi) \geq \Psi(t, \psi)\) for all \((t, \psi) \in [0, T] \times \mathbb{C}\). Let \(\tilde{\tau} \in T_t^\tau\). If \(\tilde{\tau} = t\), then by (10), we can get

\[
J(\tilde{\tau}; t, \psi) = \Psi(t, \psi).
\]

Therefore,

\[
V(t, \psi) = \sup_{\tau \in T_t^\tau} J(\tau; t, \psi) \geq J(\tilde{\tau}; t, \psi) = \Psi(t, \psi).
\]

(21)

The following dynamic programming principle (see Larssen [25]) will be used to derive our HJBVI:

\[
V(t, \psi) \geq \mathbb{E} \left[ \int_t^{t+\delta} e^{-\rho(s-t)} L(s, X_s) ds + e^{-\rho\delta} V(t+\delta, X_{t+\delta}) \right], \quad \forall \delta \geq 0.
\]

(22)

From this principle and Theorem 3.1, we have

\[
\lim_{\delta \downarrow 0} \mathbb{E} \left[ \frac{e^{-\rho\delta} V(t+\delta, X_{t+\delta}) - V(t, \psi)}{\delta} \right] = \lim_{\delta \downarrow 0} \mathbb{E} \left[ \frac{e^{-\rho\delta} V(t+\delta, X_{t+\delta}) - V(t+\delta, X_{t+\delta})}{\delta} \right]
+ \lim_{\delta \downarrow 0} \mathbb{E} \left[ V(t+\delta, X_{t+\delta}) - V(t, \psi) \right]
= -\rho V(t, \psi) + \frac{\partial V}{\partial t}(t, \psi) + AV(t, \psi)
\leq -L(t, \psi),
\]

for all \((t, \psi) \in [0, T] \times \mathbb{C}\) provided that \(V \in C^{1,2}_{lip}((0, T] \times \mathbb{C}) \cap D(S)\). That is,

\[
\frac{\partial V}{\partial t} + AV + L - \rho V \leq 0. \quad (23)
\]

From (21) and (23), it follows that

\[
\max \{ \Psi - V, \frac{\partial V}{\partial t} + AV + L - \rho V \} \leq 0. \quad (24)
\]
on \([0, T] \times \mathbb{C}\). The derivation of the inequality
\[
\max\{\Psi - V, \frac{\partial V}{\partial t} + AV + L - \rho V\} \geq 0 \quad (25)
\]
can be found in the next section.

We therefore have the following result.

**Theorem 3.2** Suppose \(V : [0, T] \times \mathbb{C}\) is the value function defined by (11) and satisfies Smoothness Conditions (i) and (ii). Then the value function \(V\) satisfies the following HJB variational inequality:
\[
\max \left\{ \Psi - V, \frac{\partial V}{\partial t} + AV + L - \rho V \right\} = 0 \quad (26)
\]
on \([0, T] \times \mathbb{C}\), and \(V(T, \psi) = \Psi(\psi), \forall \psi \in \mathbb{C}\).

Note that it is not known that the value function \(V\) satisfies the Smoothness Conditions (i) and (ii). Therefore, we need to consider viscosity solutions instead of classical solutions for HJB variational inequality (26). In fact, it will be shown that the value function is a unique viscosity solution of the HJB variational inequality (26). These results shall be given in the next section.

## 4 Viscosity Solution

In this section, we shall show that the value function \(V\) defined by (11) is actually the unique viscosity solution of the HJB variational inequality (26). First, let us define the viscosity solution of (26) as follows.

**Definition 4.1** Let \(w \in C([0, T] \times \mathbb{C})\). We say that \(w\) is a viscosity subsolution of (26) if for every \(\Gamma : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}\) satisfying Smoothness Conditions (i)-(ii) on \([0, T] \times \mathbb{C}\), we have
\[
\min \left\{ \Gamma(t, \psi) - \Psi(t, \psi), \rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - A\Gamma(t, \psi) - L(t, \psi) \right\} \leq 0
\]
at every \((t, \psi) \in [0, T] \times \mathbb{C}\) which is a local maximum of \(w - \Gamma\), with \(\Gamma(t, \psi) = w(t, \psi)\).

We say that \(w\) is a viscosity supersolution of (26) if, for every \(\Gamma : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}\) satisfying Smoothness Conditions (i)-(ii) on \([0, T] \times \mathbb{C}\), we have
\[
\min \left\{ \Gamma(t, \psi) - \Psi(t, \psi), \rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - A\Gamma(t, \psi) - L(t, \psi) \right\} \geq 0.
\]
at every \((t, \psi) \in [0, T] \times \mathbb{C}\) which is a local minimum of \(w - \Gamma\), with \(\Gamma(t, \psi) = w(t, \psi)\).

We say that \(w\) is a viscosity solution of (26) if it is a viscosity supersolution and a viscosity subsolution of (26).
As we can see in the definition, a viscosity solution must be continuous. So first we will show that the value function $V$ defined by (11) has this property. Actually, we have the following result:

**Lemma 4.2** The value function $V : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}$ defined in (11) is continuous and there exist constants $K > 0$ and $k \geq 1$ such that, for every $(t, \psi) \in [0, T] \times \mathbb{C}$, we have

$$|V(t, \psi)| \leq K(1 + \|\psi\|_2)^k. \quad (27)$$

**Proof.** It is clear that $V$ has at most polynomial growth, since $L$ and $\Phi$ have at most polynomial growth with the same $k \geq 1$ as in (9).

Let $\Xi(s) = X_s(t, \psi)$, $s \in [t, T]$, be the $\mathbb{C}$-valued solution of (4) with initial data $(t, \psi) \in [0, T] \times \mathbb{C}$. In view of Remark (v) following the proof of Theorem I.2 and in the proof of Theorem III.1 pp. 33 of Mohammed [34], the trajectory map $(t, \psi) \rightarrow X_s(t, \psi)$ from $[0, T] \times \mathbb{C}$ to $L^2(\Omega, \mathbb{C})$ is globally Lipschitz in $\psi$ uniformly with respect to $t$ on compact sets, and continuous in $t$ for fixed $\psi$. Therefore, given two $\mathbb{C}$-valued solutions $\Xi_1(s) = X_s(t_1, \psi_1)$ and $\Xi_2(s) = X_s(t_2, \psi_2)$, $s \in [t, T]$ of (4) with initial data $(t, \psi_1)$ and $(t, \psi_2)$, respectively, we have

$$E\|\Xi_1(s) - \Xi_2(s)\| \leq K\|\psi_1 - \psi_2\|_2, \quad (28)$$

where $K$ is a positive constant that depends on the Lipschitz constant in Assumptions 2.3 and $T$.

Using the Lipschitz continuity of $L, \Psi : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}$, there exists yet another constant $\Lambda > 0$ such that,

$$|J(\tau; t, \psi_1) - J(\tau; t, \psi_2)| \leq \Lambda E\|\Xi_1(\tau) - \Xi_2(\tau)\|. \quad (29)$$

Therefore, using (29) and (28), we have

$$|V(t, \psi_1) - V(t, \psi_2)| \leq \sup_{\tau \in T^T} |J(\tau; t, \psi_1) - J(\tau; t, \psi_2)|$$

$$\leq \Lambda \sup_{\tau \in T^T} E\|\Xi_1(\tau) - \Xi_2(\tau)\|$$

$$\leq \Lambda \|\psi_1 - \psi_2\|_2. \quad (30)$$

This implies the (uniform) continuity of $V(t, \psi)$ with respect to $\psi$.

We next show the continuity of $V(t, \psi)$ with respect to $t$. Let $\Xi_1(s) = X_s(t_1, \psi)$, $s \in [t_1, T]$ and $\Xi_2(s) = X_s(t_2, \psi)$, $s \in [t_2, T]$, be two $\mathbb{C}$-valued solutions of (4) with initial data $(t_1, \psi)$ and $(t_2, \psi)$, respectively.

Without loss of generality, we assume that $t_1 < t_2$. Then we can get

$$J(\tau; t_1, \psi) - J(\tau; t_2, \psi)$$
\[ \begin{align*}
&= \mathbb{E}\left[ \int_{t_1}^{t_2} e^{-\rho(t-t_1)} [L(\xi, \Xi_1(\xi))] d\xi \\
&\quad + \int_{t_2}^{\tau} e^{-\rho(t-t_2)} [L(\xi, \Xi_1(\xi)) - L(\xi, \Xi_2(\xi))] d\xi \\
&\quad + e^{-\rho(\tau-t_1)} \Psi(\tau, \Xi_1(\tau)) - e^{-\rho(\tau-t_2)} \Psi(\tau, \Xi_2(\tau)) \right].
\end{align*} \]

(31)

Therefore, there exists a constant \( \Lambda > 0 \) such that

\[ |J(\tau; t_1, \psi) - J(\tau; t_2, \psi)| \]

\[ \leq \Lambda \left( |t_1 - t_2| \mathbb{E}\|\Xi_1(\tau)\| + \mathbb{E}\|\Xi_1(\tau) - \Xi_2(\tau)\| \right). \]

(32)

Let \( \varepsilon > 0 \) be any given small constant. Using the compactness of \([0, T]\) and the uniform continuity of the trajectory map in \( t \), we know that there exists a constant \( \eta > 0 \) such that if \(|t_1 - t_2| < \eta\) then \( \mathbb{E}\|\Xi_1(s) - \Xi_2(s)\| \leq \frac{\varepsilon}{2K} \). In addition, there exists a constant \( K > 0 \) such that

\[ \mathbb{E}\left[ \sup_{s \in [t_1, T]} \|\Xi_1(s)\| \right] \leq K \quad \forall t_1 \in [0, T]. \]

Then for \(|t_1 - t_2| < \min \{ \eta, \frac{\varepsilon}{2K} \}\), we have

\[ |J(\tau; t_1, \psi) - J(\tau; t_2, \psi)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Consequently,

\[ |V(t_1, \psi) - V(t_2, \psi)| \leq \varepsilon. \]

This completes the proof. \( \square \)

Before we show that the value function is a viscosity solution of the HJB variational inequality (26), we need to prove some results related to the dynamic programming principle (22) which can also be found in [25]. The results are given next in Lemma 4.3 and Lemma 4.5.

**Lemma 4.3** Let \( \alpha, \beta \in T_t^T \) be \( G(t) \)-stopping times and \( t > 0 \) such that \( t \leq \alpha \leq \beta \) a.s.. Then we have

\[ \mathbb{E}\left[ e^{-\rho(\alpha-t)} V(\alpha, X_\alpha) \right] \geq \mathbb{E}\left[ \int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi \right] + \mathbb{E}\left[ e^{-\rho(\beta-t)} V(\beta, X_\beta) \right]. \]

(33)

**Proof.** By virtue of (10) and (11), we can get

\[ \mathbb{E}\left[ e^{-\rho(\beta-t)} V(\beta, X_\beta) + \int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi \right] \]
For each Definition 4.4 in the next lemma.

This completes the proof. \(\square\)

This implies that \(\tau\) is \(\epsilon\)-optimal if \(\epsilon \in T\). Using Lemma 4.3, we have

\[
E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} \ell(\xi, X_\xi) d\xi + e^{-\rho(\tau-t)} \Psi(\tau, X_\tau)\right] + E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi\right] = E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi + e^{-\rho(\tau-t)} \Psi(\tau, X_\tau)\right] \leq E\left[\int_{\alpha}^{\beta} e^{-\rho(\alpha-t)} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi + e^{-\rho(\alpha-t)} e^{-\rho(\tau-t)} \Psi(\tau, X_\tau)\right] = E[e^{-\rho(\alpha-t)} V(\alpha, X_\alpha)].
\]

This completes the proof. \(\square\)

Now let us give the definition of \(\epsilon\)-optimal stopping time which will be used in the next lemma.

Definition 4.4 For each \(\epsilon > 0\), a \(G(t)\)-stopping time \(\tau_\epsilon \in T^T_t\) is said to be \(\epsilon\)-optimal if

\[
0 \leq V(t, \psi) - E\left[\int_{\tau_\epsilon}^{T_t} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi + e^{-\rho(\tau-t)} V(\tau, X_\tau)\right] \leq \epsilon. \tag{34}
\]

Lemma 4.5 Let \(\theta\) be a stopping time such that \(\theta \leq \tau_\epsilon\) a.s., for any \(\epsilon > 0\), where \(\tau_\epsilon \in T^T_t\) is \(\epsilon\)-optimal. Then,

\[
V(t, \psi) = E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi + e^{-\rho(\theta-t)} V(\theta, X_\theta)\right]. \tag{35}
\]

Proof. Let \(\theta\) be a stopping time such that \(\theta \leq \tau_\epsilon\) a.s., for any \(\epsilon\)-optimal \(\tau_\epsilon \in T^T_t\). Using Lemma 4.3, we have

\[
E[e^{-\rho(\theta-t)} V(\theta, X_\theta)] \geq E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi\right] + E[e^{-\rho(\tau-t)} V(\tau, X_\tau)]. \tag{36}
\]

This implies that

\[
E[e^{-\rho(\theta-t)} V(\theta, X_\theta)] + E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi\right] \geq E\left[\int_{\alpha}^{\beta} e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi\right] + E[e^{-\rho(\tau-t)} V(\tau, X_\tau)]. \tag{37}
\]

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Note that $\tau_\epsilon$ is the $\epsilon$-optimal stopping time, then we can get

$$0 \leq V(t, \psi) - E \left[ \int_t^{\tau_\epsilon} e^{-\rho(\xi - t)} L(\xi, X_\xi) d\xi + e^{-\rho(\tau_\epsilon - t)} V(\tau_\epsilon, X_{\tau_\epsilon}) \right] \leq \epsilon. \quad (38)$$

On the other hand, by virtue of (37), we can get

$$V(t, \psi) - E \left[ e^{-\rho(\theta - t)} V(\theta, X_\theta) + \int_t^\theta e^{-\rho(\xi - t)} L(\xi, X_\xi) d\xi \right] \leq V(t, \psi) - E \left[ e^{-\rho(\tau_\epsilon - t)} V(\tau_\epsilon, X_{\tau_\epsilon}) + \int_t^{\tau_\epsilon} e^{-\rho(\xi - t)} L(\xi, X_\xi) d\xi \right]. \quad (39)$$

Thus, we can get

$$0 \leq V(t, \psi) - E \left[ \int_t^\theta e^{-\rho(\xi - t)} L(\xi, X_\xi) d\xi + e^{-\rho(\theta - t)} V(\theta, X_\theta) \right] \leq \epsilon. \quad (40)$$

Now we let $\epsilon \to 0$ in the above inequality and we can get

$$V(t, \psi) = E \left[ \int_t^\theta e^{-\rho(\xi - t)} L(\xi, X_\xi) d\xi \right] + E[e^{-\rho(\theta - t)} V(\theta, X_\theta)].$$

This completes the proof. \qed

Now we are ready to show that the value function $V$ defined by (11) is a viscosity solution of the HJBVI (26).

**Theorem 4.6** The value function $V$ is a viscosity solution of the HJB variational inequality (26).

**Proof.** We need to prove that $V$ is both a viscosity supersolution and a viscosity subsolution of (26).

First we prove that $V$ is a viscosity supersolution. Let $(t, \psi) \in [0, T] \times C$ and $\Gamma \in C^{1,2}_{lip}(I) \cap D(S)$ satisfying $\Gamma \leq V$ on the neighborhood $N(t, \psi)$ of $(t, \psi)$ with $\Gamma(\theta, \psi) = V(t, \psi)$, we want to prove the the viscosity supersolution inequality, i.e.,

$$\min \left\{ \Gamma(t, \psi) - \Psi(t, \psi), \rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \mathcal{A}\Gamma(t, \psi) - L(t, \psi) \right\} \geq 0. \quad (41)$$

We know that $V \geq \Psi$ on $N(t, \psi)$ and $\Gamma(t, \psi) = V(t, \psi)$, so we have

$$\Gamma(t, \psi) - \Psi(t, \psi) \geq 0.$$

Therefore, we just need to prove that

$$\rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \mathcal{A}\Gamma(t, \psi) - L(t, \psi) \geq 0.$$
Since \( \Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(S) \), by virtue of Theorem 3.1 pp. 78 of Mohammed [33], for \( t \leq s \leq T \), we have

\[
E[e^{-\rho(s-t)} \Gamma(s, X_s) - \Gamma(t, \psi)] = \mathbb{E} \left[ \int_t^s e^{-\rho(\xi-t)} \left( \frac{\partial \Gamma(\xi, X_\xi)}{\partial \xi} + A\Gamma(\xi, X_\xi) - \rho \Gamma(\xi, X_\xi) \right) d\xi \right].
\]

(42)

For any \( s \in [t, T] \) such that \((s, X_s) \in N(t, \psi)\), from Lemma 4.3, we can get

\[
V(t, \psi) \geq \mathbb{E} \left[ \int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi \right] + \mathbb{E} \left[ e^{-\rho(s-t)} V(s, X_s) \right].
\]

(43)

By virtue of (42), \( \Gamma \leq V \) and \( V(t, \psi) = \Gamma(t, \psi) \), we can get

\[
0 \geq \mathbb{E} \left[ \int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi \right] + \mathbb{E} \left[ e^{-\rho(s-t)} V(s, X_s) \right] - V(t, \psi)
\]

\[
0 \geq \mathbb{E} \left[ \int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi \right] + \mathbb{E} \left[ e^{-\rho(s-t)} \Gamma(s, X_s) \right] - \Gamma(t, \psi)
\]

\[
0 \geq \mathbb{E} \left[ \int_t^s e^{-\rho(\xi-t)} \left( L(\xi, X_\xi) + \frac{\partial \Gamma(\xi, X_\xi)}{\partial \xi} + A\Gamma(\xi, X_\xi) - \rho \Gamma(\xi, X_\xi) \right) d\xi \right].
\]

(44)

Dividing both sides of the above inequality by \((s-t)\), we have

\[
0 \geq \mathbb{E} \left[ \frac{1}{s-t} \int_t^s e^{-\rho(\xi-t)} \left( L(\xi, X_\xi) + \frac{\partial \Gamma(\xi, X_\xi)}{\partial \xi} + A\Gamma(\xi, X_\xi) - \rho \Gamma(\xi, X_\xi) \right) d\xi \right].
\]

(45)

Now let \( s \downarrow t \) in (45), and we obtain

\[
\frac{\partial \Gamma}{\partial t}(t, \psi) + A\Gamma(t, \psi) + L(t, \psi) - \rho \Gamma(t, \psi) \leq 0.
\]

(46)

which proves the inequality (41).

Next we want to prove that \( V \) is also a viscosity subsolution of (26). Let \((t, \psi) \in [0, T] \times \mathbb{C} \) and \( \Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(S) \) satisfying \( \Gamma \geq V \) on the neighborhood \( N(t, \psi) \) of \((t, \psi)\) with \( \Gamma(t, \psi) = V(t, \psi) \), we want to prove that

\[
\min \left\{ \Gamma(t, \psi) - \Psi(t, \psi), \rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - A\Gamma(t, \psi) - L(t, \psi) \right\} \leq 0.
\]

(47)

It is easy to get that

\[
\Gamma(t, \psi) = V(t, \psi) \geq \Psi(t, \psi).
\]
Therefore, we need to show that
\[ \rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \mathcal{A} \Gamma(t, \psi) - L(t, \psi) \leq 0. \]  
(48)

Let \( \theta \in T^T \) be a stopping time such that \( \theta \leq \tau_\epsilon \) for every \( \tau_\epsilon \), \( \epsilon \)-optimal stopping time. Using Lemma 4.5, we can get
\[ V(t, \psi) = \mathbb{E} \left[ \int_t^\theta e^{-\rho(\xi-t)} L(\xi, X_\xi) d\xi \right] + \mathbb{E} \left[ e^{-\rho(\theta-t)} V(\theta, X_\theta) \right]. \]  
(49)

Using the Dynkin’s formula (see Mohammed [33]), we have
\[ \mathbb{E} \left[ e^{-\rho(\theta-t)} \Gamma(\theta, X_\theta) \right] - \Gamma(t, \psi) = \mathbb{E} \left[ \int_t^\theta e^{-\rho(\xi-t)} \left( \frac{\partial \Gamma(\xi, X_\xi)}{\partial \xi} + \mathcal{A} \Gamma(\xi, X_\xi) - \rho \Gamma(\xi, X_\xi) \right) d\xi \right]. \]

Since \( \Gamma \geq V \) on \( N(t, \psi) \) and \( \Gamma(t, \psi) = V(t, \psi) \), for all \( \theta \) such that \( (\theta, X_\theta) \in N(t, \psi) \), we can get
\[ \mathbb{E} \left[ e^{-\rho(\theta-t)} V(\theta, X_\theta) \right] - V(t, \psi) \leq \mathbb{E} \left[ e^{-\rho(\theta-t)} \Gamma(\theta, X_\theta) \right] - \Gamma(t, \psi) \]
\[ = \mathbb{E} \left[ \int_t^\theta e^{-\rho(\xi-t)} \left( \frac{\partial \Gamma(\xi, X_\xi)}{\partial \xi} + \mathcal{A} \Gamma(\xi, X_\xi) - \rho \Gamma(\xi, X_\xi) \right) d\xi \right]. \]

Combining this with (49), the above inequality implies
\[ 0 \leq \mathbb{E} \left[ \int_t^\theta e^{-\rho(\xi-t)} \left( L(\xi, X_\xi) + \frac{\partial \Gamma(\xi, X_\xi)}{\partial \xi} + \mathcal{A} \Gamma(\xi, X_\xi) - \rho \Gamma(\xi, X_\xi) \right) d\xi \right]. \]  
(50)

Dividing (50) by \( \mathbb{E}[(\theta - t)] \) and sending \( \mathbb{E}[\theta] \to t \), we deduce
\[ \frac{\partial \Gamma}{\partial t}(t, \psi) + \mathcal{A} \Gamma(t, \psi) + L(t, \psi) - \rho \Gamma(t, \psi) \geq 0, \]  
(51)

which proves (48). Therefore, \( V \) is also a viscosity subsolution. This completes the proof of the theorem. \( \square \)

In order to prove the uniqueness we need following results. We will use the next result proven in Ekeland and Lebourg [10] and also in a general form in Stegall [41] and Bourgain [6]. The reader is also referred to Crandall et al [9] and Lions [31] for an application example of this result in a setting similar to ours.

**Lemma 4.7** Let \( \Phi \) be a bounded above and upper semicontinuous real-valued function on a closed ball \( B \) of a Banach space \( X \) which has the Radon-Nikodym property. Then for any \( \epsilon > 0 \) there exists an element \( u^* \in X^* \) with norm at most \( \epsilon \), where \( X^* \) is the dual of \( X \), such that \( \Phi + u^* \) attains its maximum on \( B \).
Note that every separable Hilbert space \((X, \| \cdot \|_X)\) satisfies the Radon-Nikodým property (see e.g. Ekeland and Lebourg [10]). In order to apply Lemma 4.7, we shall therefore restrict ourself to a subspace \(X\) of \(C = C([-r,0]; \mathbb{R}^n)\) which is a separable Hilbert space and dense in \(C\). One of the good candidates is the Sobolev space \(W^{1,2}((-r,0); \mathbb{R}^n)\), where

\[
W^{1,2}((-r,0); \mathbb{R}^n) = \{ \phi \in L^2([-r,0]; \mathbb{R}^n); \| \phi \|_{1,2} < \infty \},
\]

where \(\| \phi \|_{1,2} = \| \phi \|_2 + \| \phi' \|_2\), with \(\phi'\) being the first derivative in the sense of distribution of \(\phi\).

From the Sobolev embedding theorems (see e.g. Adams [1]), it is known that \(W^{1,2}((-r,0); \mathbb{R}^n) \subset C\) and that \(W^{1,2}((-r,0); \mathbb{R}^n)\) is dense in \(C\). For more about Sobolev spaces and corresponding results, one can refer to Adams [1].

**Theorem 4.8 (Comparison Principle)** Assume that \(V_1(t, \psi)\) and \(V_2(t, \psi)\) are both continuous with respect to the argument \((t, \psi) \in [0,T] \times C\) and are respectively viscosity subsolution and supersolution of (26) with at most a polynomial growth, i.e., there exist constants \(\Lambda > 0\) and \(k \geq 1\) such that,

\[
|V_i(t, \psi)| \leq \Lambda(1 + \|\psi\|_2)^k, \quad \text{for} \quad (t, \psi) \in [0,T] \times C, \quad i = 1, 2.
\]

Then, on every closed ball \(B\) of \(W^{1,2}((-r,0); \mathbb{R}^n)\), we have

\[
V_1(t, \psi) \leq V_2(t, \psi) \quad \text{for all} \quad (t, \psi) \in [0,T] \times B.
\]

Before we give the proof of the above theorem, we first need to do some preparation works.

Let \(V_1\) and \(V_2\) be, respectively, a viscosity subsolution and supersolution of (26). For any \(0 < \delta, \gamma < 1\) and for all \(\psi, \phi \in W^{1,2}((-r,0); \mathbb{R}^n)\) and \(t, s \in [0,T]\), define

\[
\Theta_{\delta, \gamma}(t, s, \psi, \phi) = \frac{1}{\delta} \left[ \| \psi - \phi \|_2^2 + \| \psi^0 - \phi^0 \|_2^2 + |t - s|^2 \right] + \gamma \left[ \exp(1 + \| \psi \|_2^2 + \| \psi^0 \|_2^2) + \exp(1 + \| \phi \|_2^2 + \| \phi^0 \|_2^2) \right],
\]

and

\[
\Phi_{\delta, \gamma}(t, s, \psi, \phi) = V_1(t, \psi) - V_2(s, \phi) - \Theta_{\delta, \gamma}(t, s, \psi, \phi),
\]

where \(\psi^0, \phi^0 \in W^{1,2}((-r,0); \mathbb{R}^n)\) with

\[
\psi^0(\theta) = \frac{\theta}{r} \psi(-r - \theta), \quad \phi^0(\theta) = \frac{\theta}{r} \phi(-r - \theta), \quad \forall \theta \in [-r,0].
\]

Moreover, using the polynomial growth condition for \(V_1\) and \(V_2\), we have

\[
\lim_{\| \psi \|_2 + \| \phi \|_2 \to \infty} \Phi_{\delta, \gamma}(t, s, \psi, \phi) = -\infty.
\]
The function $\Phi_{\delta\gamma}$ is a real-valued function that is bounded above and continuous on $[0, T] \times [0, T] \times W^{1,2}((-r, 0); R^n) \times W^{1,2}((-r, 0); R^n)$, since the embedding of $W^{1,2}((-r, 0); R^n)$ in $C$ is continuous. Therefore, from Lemma 4.7 (which is applicable by virtue of (55)), for any $1 > \epsilon > 0$, there exits a continuous linear functional $T_\epsilon$ in the topological dual of $W^{1,2}((-r, 0); R^n) \times W^{1,2}((-r, 0); R^n)$, with norm at most $\epsilon$, such that the function $\Phi_{\delta\gamma} + T_\epsilon$ attains it maximum in $[0, T] \times [0, T] \times B \times B$, for any closed ball $B$ of $W^{1,2}((-r, 0); R^n)$. (see Lemma 4.7.)

Let $B$ be a closed ball of $W^{1,2}((-r, 0); R^n)$ centered at 0. Denote by $(t_{\delta\gamma}, s_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})$ the maximum of $\Phi_{\delta\gamma} + T_\epsilon$ on $[0, T] \times [0, T] \times B \times B$.

Without loss of generality, we assume that for any given $\delta, \gamma$, there exists a constant $M_{\delta\gamma}$, such that the maximum value $\Phi_{\delta\gamma} + T_\epsilon + M_{\delta\gamma}$ is zero. In other words, we have

$$\Phi_{\delta\gamma}(t_{\delta\gamma}, s_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma}) + T_\epsilon(\psi_{\delta\gamma}, \phi_{\delta\gamma}) + M_{\delta\gamma} = 0. \tag{56}$$

We have the following lemmas.

Lemma 4.9 Let $B$ be a closed ball of $W^{1,2}((-r, 0); R^n)$ centered at 0, and let $(t_{\delta\gamma}, s_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})$ be the maximum of $\Phi_{\delta\gamma} + T_\epsilon + M_{\delta\gamma}$ on $[0, T] \times [0, T] \times B \times B$ for some $\delta, \gamma, \epsilon \in (0, 1)$.

Then, we have

$$\lim_{\epsilon \to 0, \delta \to 0} \left( \|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|^2_2 + \|\phi_{\delta\gamma}^{(0)} - \phi_{\delta\gamma}^{(0)}\|^2_2 + |t_{\delta\gamma} - s_{\delta\gamma}|^2 \right) = 0, \tag{57}$$

Proof. Let $r_B$ be the radius of the ball $B$. Then, for all $\delta, \gamma, \epsilon \in (0, 1)$, we have

$$\|\psi_{\delta\gamma}\|^2_2 \leq \|\psi_{\delta\gamma}\|^2_2 < r_B^2, \quad \text{and} \quad \|\phi_{\delta\gamma}\|^2_2 \leq \|\phi_{\delta\gamma}\|^2_2 < r_B^2.$$

Noting that $(t_{\delta\gamma}, s_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})$ is the maximum of $\Phi_{\delta\gamma} + T_\epsilon + M_{\delta\gamma}$, we get

$$\Phi_{\delta\gamma}(t_{\delta\gamma}, t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma}) + T_\epsilon(\psi_{\delta\gamma}, \phi_{\delta\gamma}) + \Phi_{\delta\gamma}(s_{\delta\gamma}, s_{\delta\gamma}, \phi_{\delta\gamma}, \phi_{\delta\gamma}) + T_\epsilon(\phi_{\delta\gamma}, \phi_{\delta\gamma}) \leq 2\Phi_{\delta\gamma}(t_{\delta\gamma}, s_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma}) + 2T_\epsilon(\psi_{\delta\gamma}, \phi_{\delta\gamma}). \tag{58}$$

It implies

$$V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - V_2(t_{\delta\gamma}, \psi_{\delta\gamma}) - 2\gamma(\exp(1 + \|\psi_{\delta\gamma}\|^2_2 + \|\phi_{\delta\gamma}^{(0)}\|^2_2))$$

$$+ T_\epsilon(\psi_{\delta\gamma}, \phi_{\delta\gamma}) + V_1(s_{\delta\gamma}, \phi_{\delta\gamma}) - V_2(s_{\delta\gamma}, \phi_{\delta\gamma}) - 2\gamma(\exp(1 + \|\phi_{\delta\gamma}\|^2_2 + \|\phi_{\delta\gamma}^{(0)}\|^2_2)) + T_\epsilon(\phi_{\delta\gamma}, \phi_{\delta\gamma})$$

$$\leq 2V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - 2V_2(s_{\delta\gamma}, \phi_{\delta\gamma}) - \frac{2\gamma}{\delta} \left[ \|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|^2_2 + \|\phi_{\delta\gamma}^{(0)} - \phi_{\delta\gamma}^{(0)}\|^2_2 ight]$$

$$+ |t_{\delta\gamma} - s_{\delta\gamma}|^2 - 2\gamma \left( \exp(1 + \|\psi_{\delta\gamma}\|^2_2 + \|\phi_{\delta\gamma}^{(0)}\|^2_2) ight)$$

$$+ \exp(1 + \|\phi_{\delta\gamma}\|^2_2 + \|\phi_{\delta\gamma}^{(0)}\|^2_2)) + 2T_\epsilon(\psi_{\delta\gamma}, \phi_{\delta\gamma}). \tag{59}$$
From the above inequality, it is easy to get that
\[
\frac{2}{\delta} \left[ \left\| \psi_{\delta}\gamma_\epsilon - \phi_{\delta}\gamma_\epsilon \right\|^2 + \left\| \psi^0_{\delta}\gamma_\epsilon - \phi^0_{\delta}\gamma_\epsilon \right\|^2 + |t_{\delta}\gamma_\epsilon - s_{\delta}\gamma_\epsilon|^2 \right]
\leq [V_1(t_{\delta}\gamma_\epsilon, \psi_{\delta}\gamma_\epsilon) - V_1(s_{\delta}\gamma_\epsilon, \phi_{\delta}\gamma_\epsilon)] + [V_2(t_{\delta}\gamma_\epsilon, \psi_{\delta}\gamma_\epsilon) - V_2(s_{\delta}\gamma_\epsilon, \phi_{\delta}\gamma_\epsilon)]
+ 2T_\epsilon(\psi_{\delta}\gamma_\epsilon, \phi_{\delta}\gamma_\epsilon) - [T_\epsilon(\psi_{\delta}\gamma_\epsilon, \psi_{\delta}\gamma_\epsilon) + T_\epsilon(\phi_{\delta}\gamma_\epsilon, \phi_{\delta}\gamma_\epsilon)].
\] (60)

From the polynomial growth condition of $V_1, V_2$, and the fact that the norm of $T_\epsilon$ is $\epsilon \in (0, 1)$, we can find a constant $\Lambda$ and a positive integer $k \geq 2$ such that
\[
\frac{2}{\delta} \left[ \left\| \psi_{\delta}\gamma_\epsilon - \phi_{\delta}\gamma_\epsilon \right\|^2 + \left\| \psi^0_{\delta}\gamma_\epsilon - \phi^0_{\delta}\gamma_\epsilon \right\|^2 + |t_{\delta}\gamma_\epsilon - s_{\delta}\gamma_\epsilon|^2 \right]
\leq \Lambda (1 + \|\psi_{\delta}\gamma_\epsilon\|_2 + \|\phi_{\delta}\gamma_\epsilon\|_2)^k \leq \Lambda (1 + 2r)^k.
\] (61)

So
\[
\left\| \psi_{\delta}\gamma_\epsilon - \phi_{\delta}\gamma_\epsilon \right\|^2 + \left\| \psi^0_{\delta}\gamma_\epsilon - \phi^0_{\delta}\gamma_\epsilon \right\|^2 + |t_{\delta}\gamma_\epsilon - s_{\delta}\gamma_\epsilon|^2
\leq \delta \Lambda (1 + 2r)^k,
\] (62)

In order to obtain (57), we send $\delta$ to zero in (62) using the above inequality. \(\square\)

Now let us introduce a functional $F : W^{1,2}((-r,0); \mathbb{R}^n) \rightarrow \mathbb{R}$ defined by
\[
F(\psi) \equiv \|\psi\|^2_2.
\] (63)

and the linear map $H : W^{1,2}((-r,0); \mathbb{R}^n) \rightarrow \mathbb{C}$ defined by
\[
H(\psi)(\theta) \equiv \frac{\theta}{\epsilon} \psi(-r - \theta) = \psi^0(\theta), \quad \theta \in [-r, 0].
\] (64)

It is easy to verify that
\[
H(\psi)(0) = \psi^0(0) = 0, \quad H(\psi)(-r) = \psi^0(-r) = -\psi(0).
\] (65)

$H$ is continuous because $\|H(\psi)\| \leq \|\psi\| \leq K \|\psi\|_1$ for some $K > 0$, where the last inequality comes from the Sobolev embedding inequality.

It is not hard to show that the map $F$ is Fréchet differentiable and its derivative is given by $DF(u)h = 2(u|h)$. This comes from the fact that
\[
\|\psi + h\|^2_2 - \|\psi\|^2_2 = 2(\psi|h) + \|h\|^2_2,
\]
and we can always find a constant $\Lambda > 0$ such that
\[
\frac{\|\psi + h\|^2_2 - \|\psi\|^2_2 - 2(\psi|h)}{\|h\|_1} \leq \frac{\|h\|^2_2}{\|h\|_2} \leq \frac{\Lambda \|h\|^2_1}{\|h\|_2} = \Lambda \|h\|_1.
\] (66)

Moreover, we have
\[
2(\psi + h \cdot) - 2(\psi \cdot) = 2(h \cdot).
\]
We deduce that $F$ is twice differentiable and $D^2 F(u)(h,k) = 2(h|k)$.

In addition, the map $H$ is linear and twice continuously Fréchet differentiable. Therefore, $DH(\psi)(h) = H(h)$ and $D^2 H(\psi)(h,k) = 0$, for all $\psi, h, k \in W^{1,2}((-r,0); \mathbb{R}^n)$.

From the definition of $\Theta_{\delta \gamma}$ and the definition of $F$, we can get that

$$
\Theta_{\delta \gamma}(t,s,\psi,\phi) = \frac{1}{\delta} \left[ F(\psi - \phi) + F(H(\psi) - H(\phi)) + |t - s|^2 \right] + \gamma \left[ e^{1+F(\psi)+F(H(\psi))} + e^{1+F(\phi)+F(H(\phi))} \right].
$$

The following chain rule, quoted from [40] (Theorem 5.2.5 on page 208), is needed to get the Fréchet derivatives of $\Theta_{\delta \gamma}$:

**Theorem 4.10 (Chain Rule)** Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be real Banach spaces. If $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{Z}$ are Fréchet differentiable at $x$ and $y = S(x) \in \mathcal{Y}$, respectively, then $U = T \circ S$ is Fréchet differentiable at $x$ and its Fréchet derivative is given by

$$
D_x U(x) = D_y T(S(x)) D_x S(x).
$$

Given the above chain rule, we can say that $\Theta_{\delta \gamma}$ is Fréchet differentiable. Actually, for $h, k \in W^{1,2}((-r,0); \mathbb{R}^n)$, we can get

$$
D_{\psi} \Theta_{\delta \gamma}(t,s,\psi,\phi)(h) = \frac{2}{\delta} \left[ (\psi - \phi|h) + (H(\psi - \phi)|H(h)) \right] + 2\gamma e^{1+F(\psi)+F(H(\psi))} \left[ (\psi|h) + (H(\psi)|H(h)) \right]. \quad (67)
$$

Similarly,

$$
D_{\phi} \Theta_{\delta \gamma}(t,s,\psi,\phi)(k) = \frac{2}{\delta} \left[ (\phi - \psi|k) + (H(\phi - \psi)|H(k)) \right] + 2\gamma e^{1+F(\psi)+F(H(\psi))} \left[ (\phi|k) + (H(\phi)|H(k)) \right]. \quad (68)
$$

Furthermore,

$$
D^2_{\psi} \Theta_{\delta \gamma}(t,s,\psi,\phi)(h,k) = \frac{2}{\delta} \left[ (h|k) + (H(h)|H(k)) \right] + 2\gamma e^{1+F(\psi)+F(H(\psi))} \left[ 2((\psi|h) + (H(\psi)|H(h)))(\psi|h) + (H(\psi)|H(h)) \right] + (k|h) + (H(k)|H(h)). \quad (69)
$$
Similarly,
\[ D^2 \Theta_{\delta \gamma}(t, s, \psi, \phi)(h, k) \]
\[ = 2 \delta \left[ (h|k) + (H(h)|H(k)) \right] \]
\[ + 2 \gamma e^{1+F(\psi) + F(H(\psi))} \left[ 2((\phi|h) + (H(\phi)|H(k))((\phi|h) + (H(\phi)|H(h))) \right] \]
\[ + (k|h) + (H(k)|H(h)) \right]. \tag{70} \]

Observe that we can extend \( D^2 \Theta_{\delta \gamma}(t, s, \psi, \phi) \) and \( D^2 \Theta_{\delta \gamma}(t, s, \psi, \phi) \), the first and second order Fréchet derivatives of \( \Theta_{\delta \gamma} \) with respect to \( \psi \), to the space \( W^{1,2}((-r, 0); \mathbb{R}^n) \oplus B \) (see Section 3 of this paper or Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [33]) by setting
\[ D^2 \Theta_{\delta \gamma}(t, s, \psi, \phi)(h + v \mathbf{1}_{(0)}) \]
\[ = 2 \delta \left[ (\psi - \phi|h + v \mathbf{1}_{(0)}) + (H(\psi - \phi)|H(h + v \mathbf{1}_{(0)})) \right] \]
\[ + 2 \gamma e^{1+F(\psi) + F(H(\psi))} \left[ 2((\psi|h + v \mathbf{1}_{(0)}) + (H(\psi)|H(h + v \mathbf{1}_{(0)}))) \right] \]
\[ \cdot (\psi|h + v \mathbf{1}_{(0)}) + (H(\psi)|H(h + v \mathbf{1}_{(0)})) \]
\[ + (k + w \mathbf{1}_{(0)}|h + v \mathbf{1}_{(0)}) + (H(k + w \mathbf{1}_{(0)})|H(h + v \mathbf{1}_{(0)})) \right], \tag{71} \]

and
\[ D^2 \Theta_{\delta \gamma}(t, s, \psi, \phi)(h + v \mathbf{1}_{(0)}, k + w \mathbf{1}_{(0)}) \]
\[ = 2 \delta \left[ (h + v \mathbf{1}_{(0)})|k + w \mathbf{1}_{(0)}) + (H(h + v \mathbf{1}_{(0)})|H(k + w \mathbf{1}_{(0)})) \right] \]
\[ + 2 \gamma e^{1+F(\psi) + F(H(\psi))} \left[ 2((\psi|h + v \mathbf{1}_{(0)}) + (H(\psi)|H(h + v \mathbf{1}_{(0)}))) \right] \]
\[ \cdot (\psi|h + v \mathbf{1}_{(0)}) + (H(\psi)|H(h + v \mathbf{1}_{(0)})) \]
\[ + (k + w \mathbf{1}_{(0)}|h + v \mathbf{1}_{(0)}) + (H(k + w \mathbf{1}_{(0)})|H(h + v \mathbf{1}_{(0)})) \right], \tag{72} \]

for \( v, w \in \mathbb{R}^n \) and \( h, k \in W^{1,2}((-r, 0); \mathbb{R}^n) \).

Moreover, it is easy to see that these extensions are continuous for that there exists a constant \( \Lambda > 0 \) such that
\[ |(\psi - \phi|h + v \mathbf{1}_{(0)})| \leq \|\psi - \phi\|_2 \cdot \|h + v \mathbf{1}_{(0)}\|_2 \leq \Lambda\|\psi - \phi\|_2(\|h\| + |v|); \tag{73} \]
\[ |(\psi|h + v \mathbf{1}_{(0)})| \leq \|\psi\|_2 \cdot \|h + v \mathbf{1}_{(0)}\|_2 \leq \Lambda\|\psi\|_2(\|h\| + |v|); \tag{74} \]
\[ |(\psi|k + w \mathbf{1}_{(0)})| \leq \|\psi\|_2 \cdot \|k + w \mathbf{1}_{(0)}\|_2 \leq \Lambda\|\psi\|_2(\|k\| + |w|); \tag{75} \]

and
\[ |(k + w \mathbf{1}_{(0)}|h + v \mathbf{1}_{(0)}| \leq \|k + w \mathbf{1}_{(0)}\|_2\|h + v \mathbf{1}_{(0)}\|_2 \leq \Lambda(\|k\| + |w|)(\|h\| + |v|). \tag{76} \]
Similarly, we can extend the first and second order Fréchet derivatives of $\Theta_{\delta \gamma}$ with respect to $\phi$, to the space $W^{1,2}((−r,0); \mathbb{R}^n) \oplus B$ and obtain similar expressions for $D_\phi \Theta_{\delta \gamma}(t, s, \psi, \phi)(k+w1_{(0)})$ and $D^2_\phi \Theta_{\delta \gamma}(t, s, \psi, \phi)(h+v1_{(0)}, k+w1_{(0)})$.

The same is also true for the bounded linear functional $T_{\epsilon}$ whose extension is still written as $T_{\epsilon}$.

In addition, it is easy to verify that for any $\phi \in W^{1,2}((−r,0); \mathbb{R}^n)$ and $v, w \in \mathbb{R}^n$ we have

\[
(\phi|v1_{(0)}) = \int_{−r}^{0} \langle \phi(s), v1_{(0)}(s) \rangle ds = 0, \tag{77}
\]

\[
(w1_{(0)}|v1_{(0)}) = \int_{−r}^{0} \langle w1_{(0)}(s), v1_{(0)}(s) \rangle ds = 0, \tag{78}
\]

\[
H(v1_{(0)}) = v1_{(−r)}, \tag{79}
\]

\[
H(\psi)|H(v1_{(0)}) = 0, \quad (H(w1_{(0)})|H(v1_{(0)})) = 0. \tag{80}
\]

These observations shall be used later on. Next we need several lemmas about the operator $S$.

**Lemma 4.11** Given $\phi \in W^{1,2}((−r,0); \mathbb{R}^n)$, we have

\[
S(F)(\phi) = |\phi(0)|^2 - |\phi(−r)|^2, \tag{81}
\]

\[
S(F)(\phi^0) = −|\phi(0)|^2, \tag{82}
\]

where $F$ is the functional defined in (63) and $S$ is the operator defined in (15). Moreover $S(F)$ is continuous from $W^{1,2}((−r,0); \mathbb{R}^n)$ to $\mathbb{R}$.

**Proof.** Recall that

\[
S(F)(\phi) = \lim_{t \downarrow 0} \frac{1}{t} \left[ F(\tilde{\phi}_t) - F(\phi) \right] \tag{83}
\]

for all $\phi \in W^{1,2}((−r,0); \mathbb{R}^n)$, where $\tilde{\phi} : [−r,T] \rightarrow \mathbb{R}^n$ is an extension of $\phi$ defined by

\[
\tilde{\phi}(t) = \begin{cases} 
\phi(t) & \text{if } t \in [−r,0) \\
\phi(0) & \text{if } t \geq 0,
\end{cases} \tag{84}
\]

and again $\tilde{\phi}_t \in W^{1,2}((−r,0); \mathbb{R}^n)$ is defined by

\[
\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [−r,0].
\]

Therefore, we have

\[
S(F)(\phi) = \lim_{t \downarrow 0} \frac{1}{t} \left[ \|\tilde{\phi}_t\|_2^2 - \|\phi\|_2^2 \right] = \lim_{t \downarrow 0} \frac{1}{t} \left[ \int_{−r}^{0} |\tilde{\phi}_t(\theta)|^2 d\theta - \int_{−r}^{0} |\phi(\theta)|^2 d\theta \right]
\]

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Define an operator \( \tilde{\psi} \)

\[
\begin{align*}
&= \lim_{t \to 0} \frac{1}{t} \left[ \int_{-r}^{0} |\tilde{\psi}(\theta + t)|^2 d\theta - \int_{-r}^{0} |\phi(\theta)|^2 d\theta \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[ \int_{-r}^{0} |\tilde{\psi}(\theta)|^2 d\theta - \int_{-r}^{0} |\phi(\theta)|^2 d\theta \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[ \int_{-r}^{0} |\tilde{\psi}(\theta)|^2 d\theta + \int_{0}^{t} |\tilde{\psi}(\theta)|^2 d\theta - \int_{-r}^{0} |\phi(\theta)|^2 d\theta \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[ \int_{-r}^{0} |\phi(\theta)|^2 d\theta - \int_{-r}^{0} |\phi(\theta)|^2 d\theta \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[ \int_{0}^{t} |\phi(\theta)|^2 d\theta - \int_{0}^{t} |\phi(\theta)|^2 d\theta \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[ \int_{-r}^{0} |\phi(\theta)|^2 d\theta - \int_{-r}^{0} |\phi(\theta)|^2 d\theta \right] \\
&= \left| \phi(0) \right|^2 - \left| \phi(-r) \right|^2. \tag{85}
\end{align*}
\]

Similarly, we have

\[
S(F)(\phi^0) = |\phi^0(0)|^2 - |\phi^0(-r)|^2 = -|\phi(0)|^2.
\]

It is easy to verify that

\[
|K(\psi) - K(\phi)| \leq |\psi(0) - \phi(0)| \leq \|\psi - \phi\| \leq \Lambda \|\psi - \phi\|_{1,2} \quad \text{for some} \quad \Lambda > 0.
\]

Therefore, the map \( K : W^{1,2}((-r, 0); \mathbb{R}^n) \to \mathbb{R} \) defined by \( K(\psi) = |\psi(0)| \) is continuous.

Similarly the map \( J : W^{1,2}((-r, 0); \mathbb{R}^n) \to \mathbb{R} \) defined by \( J(\psi) = |\psi(-r)| \) is continuous. Therefore \( S(F) \) is continuous because \( S(F)(\psi) = K(\psi)^2 - J(\psi)^2, \) for all \( \psi \in W^{1,2}((-r, 0); \mathbb{R}^n) \).

Let \( S_{\psi} \) and \( S_{\phi} \) denote the operator \( S \) applied to \( \psi \) and \( \phi \), respectively. We have the following lemma:

**Lemma 4.12** Given \( \phi, \psi \in W^{1,2}((-r, 0); \mathbb{R}^n), \)

\[
S_{\psi}(F)(\phi - \psi) + S_{\phi}(F)(\phi - \psi) = |\psi(0) - \phi(0)|^2 - |\psi(-r) - \phi(-r)|^2, \tag{86}
\]

and

\[
S_{\psi}(F)(\phi^0 - \psi^0) + S_{\phi}(F)(\phi^0 - \psi^0) = -|\psi(0) - \phi(0)|^2, \tag{87}
\]

where \( F \) is the functional defined in (63) and \( S \) is the operator defined in (15).

**Proof.** Define an operator \( \tilde{S} : D(\tilde{S}) \to W^{1,2}((-r, 0); \mathbb{R}^n) \) is defined by

\[
\tilde{S}(\psi) = \lim_{t \to 0} \frac{\tilde{\psi}_t - \psi}{t}. \tag{88}
\]
where the domain $\mathcal{D}(\tilde{S})$ consists of those $\psi \in W^{1,2}((-r, 0); \mathbb{R}^n)$ for which the above limit exists.

It is easy to see that the domain $\mathcal{D}(\tilde{S})$ of $\tilde{S}$ contains differential functions of $W^{1,2}((-r, 0); \mathbb{R}^n)$, therefore $\mathcal{D}(\tilde{S})$ is dense in $W^{1,2}((-r, 0); \mathbb{R}^n)$.

We first assume that $\psi \in \mathcal{D}(\tilde{S})$. By the definition of $S$ and $F$, we have

$$\mathcal{S}(F)(\psi) = \lim_{t \downarrow 0} \frac{1}{t} \left[ F(\tilde{\psi}_t) - F(\psi) \right]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ (\tilde{\psi}_t|\tilde{\psi}_t) - (\psi|\psi) \right]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ (\tilde{\psi}_t|\tilde{\psi}_t - \psi) + (\tilde{\psi}_t - \psi|\psi) \right]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ (\tilde{\psi}_t + \psi|\tilde{\psi}_t - \psi) \right]$$

$$= \left( \lim_{t \downarrow 0} \left[ \tilde{\psi}_t + \psi \right] \right) \lim_{t \downarrow 0} \frac{1}{t} \left[ \tilde{\psi}_t - \psi \right]$$

$$= 2(\psi|\tilde{S}(\psi)). \quad (89)$$

On the other hand, by virtue of Lemma 4.11, we have

$$\mathcal{S}(F)(\psi) = |\psi(0)|^2 - |\psi(-r)|^2. \quad (90)$$

Therefore, we have

$$(\psi|\tilde{S}(\psi)) = \frac{1}{2} \left[ |\psi(0)|^2 - |\psi(-r)|^2 \right]. \quad (91)$$

Since $\tilde{S}$ is a linear operator, we have

$$(\psi - \phi|\tilde{S}(\psi) - \tilde{S}(\phi)) = (\psi - \phi|\tilde{S}(\psi - \phi))$$

$$= \frac{1}{2} \left[ |\psi(0) - \phi(0)|^2 - |\psi(-r) - \phi(-r)|^2 \right]. \quad (92)$$

Given the above results, now we have

$$\mathcal{S}_\psi(F)(\psi - \phi) + \mathcal{S}_\phi(F)(\psi - \phi)$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ ||\tilde{\psi}_t - \phi||^2 - ||\psi - \phi||^2 + ||\psi - \tilde{\phi}_t||^2 - ||\psi - \phi||^2 \right]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ ||\tilde{\psi}_t||^2 - ||\psi||^2 + ||\tilde{\phi}_t||^2 - ||\phi||^2 \right]$$

$$- 2((\tilde{\psi}_t|\phi) - (\psi|\phi) + (\psi|\tilde{\phi}_t) - (\psi|\phi))$$
\[ S(F)(\psi) + S(F)(\phi) - 2[(\tilde{S}(\psi)|\phi) + (\psi|\tilde{S}(\phi))] \\
= 2(\psi|\tilde{S}(\psi)) + 2(\phi|\tilde{S}(\phi)) - 2[(\tilde{S}(\psi)|\phi) + (\psi|\tilde{S}(\phi))] \\
= 2(\psi - \phi|\tilde{S}(\psi - \phi)) \\
= [\psi(0) - \phi(0)]^2 - [\psi(-r) - \phi(-r)]^2, \]

provided that \( \psi, \phi \in D(\tilde{S}) \).

For any \( \psi, \phi \in W^{1,2}((-r, 0); \mathbb{R}^n) \), one can construct sequences \( \{\psi_k\}_{k=1}^\infty \) and \( \{\phi_k\}_{k=1}^\infty \) in \( D(\tilde{S}) \) such that

\[
\lim_{k \to \infty} \|\psi_k - \psi\|_{1,2} = 0 \quad \text{and} \quad \lim_{k \to \infty} \|\phi_k - \phi\|_{1,2} = 0. 
\]

Consequently, by the continuity the \( S(F) \) operator, we have

\[
S_\psi(F)(\psi - \phi) + S_\phi(F)(\psi - \phi) \\
= \lim_{k \to \infty} \left( S_\psi(F)(\psi_k - \phi_k) + S_\phi(F)(\psi_k - \phi_k) \right) \\
= \lim_{k \to \infty} \|\psi_k(0) - \phi_k(0)\|^2 - |\psi_k(-r) - \phi_k(-r)|^2 \\
= [\|\psi(0) - \phi(0)\|^2 - |\psi(-r) - \phi(-r)|^2].
\]

By the same argument, we can get

\[
S_\psi(F)(\psi_0 - \phi_0) + S_\phi(F)(\psi_0 - \phi_0) \\
= |\psi_0(0) - \phi_0(0)|^2 - |\psi_0(-r) - \phi_0(-r)|^2 \\
= -|\psi(0) - \phi(0)|^2.
\]

\[ \square \]

**Lemma 4.13** Given \( \phi \in W^{1,2}((-r, 0); \mathbb{R}^n) \), we define a new operator \( G \) as follows

\[ G(\phi) = e^{1+F(\phi)+F(\phi^0)}. \] (93)

Then we have,

\[ S(G)(\phi) = (-|\phi(-r)|^2)e^{1+F(\phi)+F(\phi^0)}, \] (94)

where \( F \) is the functional defined in (63) and \( S \) is the operator defined in (15).

**Proof.** Recall that

\[ S(G)(\phi) = \lim_{t \to 0} \frac{1}{t} \left[ G(\tilde{\phi}_t) - G(\phi) \right] \] (95)

for all \( \phi \in W^{1,2}((-r, 0); \mathbb{R}^n) \), where \( \tilde{\phi} : [-r, T] \to \mathbb{R}^n \) is an extension of \( \phi \) defined by

\[ \tilde{\phi}(t) = \begin{cases} 
\phi(t) & \text{if } t \in [-r, 0) \\
\phi(0) & \text{if } t \geq 0,
\end{cases} \] (96)
and again $\tilde{\phi}_t \in W^{1,2}((-r, 0); \mathbb{R}^n)$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-r, 0].$$

Then we have

$$S(G)(\phi) = \lim_{t \downarrow 0} \frac{1}{t} \left[ e^{1 + f_{r+}^0} \tilde{\phi}_t(\theta)^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \right. - e^{1 + f_{r+}^0} \tilde{\phi}_{t+}(\theta)^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \biggl]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ e^{1 + f_{r+}^0} \tilde{\phi}_t(\theta)^2 d\theta + f_{r+}^0 |\phi(\theta+t)|^2 d\theta \right. - e^{1 + f_{r+}^0} \tilde{\phi}_{t+}(\theta)^2 d\theta + f_{r+}^0 |\phi(\theta+t)|^2 d\theta \biggl]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ e^{1 + f_{r+}^0} |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \right. - e^{1 + f_{r+}^0} |\phi(\theta+t)|^2 d\theta + f_{r+}^0 |\phi(\theta+t)|^2 d\theta \biggl]$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \left[ e^{1 + f_{r+}^0} |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \right. - e^{1 + f_{r+}^0} |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \biggl].$$

Using the L’Hospital rule on the last equality and (65), we obtain

$$S(G)(\phi) = \lim_{t \downarrow 0} \frac{1}{t} \left[ e^{1 + f_{r+}^0} |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \right. - e^{1 + f_{r+}^0} |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta \biggl]$$

$$= (|\phi(0)|^2 - |\phi(-r)|^2 - |\phi(0)|^2 - |\phi(0)|^2) e^{1 + f_{r+}^0} |\phi(\theta)|^2 d\theta + f_{r+}^0 |\phi(\theta)|^2 d\theta$$

$$= -|\phi(-r)|^2 e^{1 + F(\phi) + F(\phi^0)}.$$

which completes the proof. □

Given all above results, now we are ready to prove Theorem 4.8.

**Proof of Theorem 4.8.** Let $B$ a closed ball of $W^{1,2}((-r, 0); \mathbb{R}^n)$. For all $s, t \in [0, T]$ and $\psi, \phi \in B$, and for any given $\gamma, \delta, \epsilon$, we define

$$\Gamma_1(t, \psi) \equiv V_2(s_{\delta \gamma}, \phi_{\delta \gamma}) + \Theta_{\delta \gamma}(t, s_{\delta \gamma}, \psi, \phi_{\delta \gamma}) - T_\epsilon(s_{\delta \gamma}, \phi_{\delta \gamma}) - M_{\delta \gamma},$$

(99)
and

$$\Gamma_2(s, \phi) \equiv V_1(t_{\delta \gamma, \psi_{\delta \gamma}}) - \Theta_{\delta \gamma}(t_{\delta \gamma, s, \psi_{\delta \gamma}, \phi}) + T_\epsilon(\psi_{\delta \gamma}, \phi_{\delta \gamma}) + M_{\delta \gamma}$$  \hspace{1cm} (100)$$

where $$(t_{\delta \gamma, s, \psi_{\delta \gamma}, \phi_{\delta \gamma}})$$ is maximum point of $$\Phi_{\delta \gamma} + T_\epsilon + M_{\delta \gamma}$$ over $$[0, T] \times [0, T] \times B \times B$$.

Recall that

$$\Phi_{\delta \gamma}(t, s, \psi, \phi) = V_1(t, \psi) - V_2(s, \phi) - \Theta_{\delta \gamma}(t, s, \psi, \phi),$$

and $$\Phi_{\delta \gamma} + T_\epsilon + M_{\delta \gamma}$$ reaches its maximum value zero at $$(t_{\delta \gamma, s, \psi_{\delta \gamma}, \phi_{\delta \gamma}})$$ over $$[0, T] \times [0, T] \times B \times B$$.

By the definition of $$\Gamma_1$$ and $$\Gamma_2$$, it is easy to verify that

$$\Gamma_1(t, \psi) \geq V_1(t, \psi), \quad \Gamma_2(s, \phi) \leq V_2(s, \phi), \quad \forall t, s \in [0, T] \text{ and } \phi, \psi \in B,$$

and

$$V_1(t_{\delta \gamma, \psi_{\delta \gamma}}) = \Gamma_1(t_{\delta \gamma, \psi_{\delta \gamma}}) \text{ and } V_2(s_{\delta \gamma, \phi_{\delta \gamma}}) = \Gamma_2(s_{\delta \gamma, \phi_{\delta \gamma}}).$$

Using the definitions of the viscosity subsolution, $$V_1$$ and $$\Gamma_1$$, we have

$$\min \left\{ V_1(t_{\delta \gamma, \psi_{\delta \gamma}}) - \Psi(t, \psi_{\delta \gamma}), \right.$$ \hspace{1cm} (101)

$$\left. \rho V_1(t_{\delta \gamma, \psi_{\delta \gamma}}) - \frac{\partial \Gamma_1}{\partial t}(t_{\delta \gamma, \psi_{\delta \gamma}}) - A(\Gamma_1)(t_{\delta \gamma, \psi_{\delta \gamma}}) - L(t_{\delta \gamma, \psi_{\delta \gamma}}) \right\} \leq 0.$$

By the definitions of the operator $$A$$ and $$\Gamma_1$$, and by virtue of (67), (68), (69), (70), (71), (72), (77), (78), and (80), we can get

$$A(\Gamma_1)(t_{\delta \gamma, \psi_{\delta \gamma}}) = S(\Gamma_1)(t_{\delta \gamma, \psi_{\delta \gamma}}) + D_{\psi} \Theta_{\delta \gamma}(\cdot \cdots)(f(t_{\delta \gamma, \psi_{\delta \gamma}})\mathbf{1}_{\{0\}})$$

$$+ \frac{1}{2} \sum_{j=1}^{m} D_{\psi}^2 \Theta_{\delta \gamma}(\cdot \cdots)(g(t_{\delta \gamma, \psi_{\delta \gamma}})(\mathbf{e}_j)\mathbf{1}_{\{0\}}) \cdot g(t_{\delta \gamma, \psi_{\delta \gamma}})(\mathbf{e}_j)\mathbf{1}_{\{0\}}$$

$$= S(\Gamma_1)(t_{\delta \gamma, \psi_{\delta \gamma}}).$$

Note that $$\Theta_{\delta \gamma}(\cdot \cdots)$$ is an abbreviation for $$\Theta_{\delta \gamma}(t_{\delta \gamma, s_{\delta \gamma}, \psi_{\delta \gamma}, \phi_{\delta \gamma}})$$ in the above equation and the following.

It follows from (101) and the above inequality together that

$$\min \left\{ V_1(t_{\delta \gamma, \psi_{\delta \gamma}}) - \Psi(t_{\delta \gamma, \psi_{\delta \gamma}}), \right.$$ \hspace{1cm} (102)

$$\rho V_1(t_{\delta \gamma, \psi_{\delta \gamma}}) - S(\Gamma_1)(t_{\delta \gamma, \psi_{\delta \gamma}}) - \frac{\partial \Gamma_1}{\partial t}(t_{\delta \gamma, \psi_{\delta \gamma}}) - L(t_{\delta \gamma, \psi_{\delta \gamma}}) \right\} \leq 0.$$
Similarly, using the definitions of the viscosity supersolution of $V_2$ and $\Gamma_2$ and by the virtue of the same techniques similar to (102), we have

\[
\min \left\{ V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) - \Psi(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}), \right.
\rho V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) - \mathcal{S}(\Gamma_2)(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) - \frac{\partial \Gamma_2}{\partial s}(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon})
\left. - L(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) \right\} \geq 0.
\]

(103)

By virtue of (99), (100) and (53), we can get

\[
\frac{\partial \Gamma_1}{\partial t}(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) = \frac{2}{\delta}(t_{\delta\gamma\epsilon} - s_{\delta\gamma\epsilon}),
\]

(104)

\[
\frac{\partial \Gamma_2}{\partial s}(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) = -\frac{2}{\delta}(s_{\delta\gamma\epsilon} - t_{\delta\gamma\epsilon}) = \frac{2}{\delta}(t_{\delta\gamma\epsilon} - s_{\delta\gamma\epsilon}),
\]

(105)

Therefore, the inequality (102) is equivalent to

\[
V_1(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - \Psi(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) \leq 0,
\]

(106)

or

\[
\rho V_1(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - \mathcal{S}(\Gamma_1)(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - \frac{2}{\delta}(t_{\delta\gamma\epsilon} - s_{\delta\gamma\epsilon}) \leq 0.
\]

(107)

Similarly, by virtue of (105), the inequality (103) is equivalent to

\[
V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) - \Psi(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) \geq 0,
\]

(108)

and

\[
\rho V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) - \mathcal{S}(\Gamma_2)(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) - \frac{2}{\delta}(s_{\delta\gamma\epsilon} - t_{\delta\gamma\epsilon}) - L(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) \geq 0.
\]

(109)

If (106) holds, using (108), we can get that there exists a constant $\Lambda > 0$ such that

\[
V_1(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon})
\leq \Psi(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - \Psi(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon})
\leq \Lambda(\|t_{\delta\gamma\epsilon} - s_{\delta\gamma\epsilon}\| + \|\psi_{\delta\gamma\epsilon} - \phi_{\delta\gamma\epsilon}\|).
\]

(110)

Thus applying Lemma 4.9 to (110), we have

\[
\limsup_{\delta \downarrow 0, \epsilon \downarrow 0}(V_1(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon})) \leq 0.
\]

(111)

On the other hand, if (107) holds, then by virtue of (107) and (109), we can obtain

\[
\rho(V_1(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - V_2(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}))
\leq \mathcal{S}(\Gamma_1)(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - \mathcal{S}(\Gamma_2)(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon})
\]

+ \left[ L(t_{\delta\gamma\epsilon}, \psi_{\delta\gamma\epsilon}) - L(s_{\delta\gamma\epsilon}, \phi_{\delta\gamma\epsilon}) \right]
\]

(112)
From the definition (15) of $S$, it is clear that $S$ is linear and takes zero on constants.

Recall that
\[
\Gamma_1(t, \psi) = V_2(s, \delta \gamma, \phi) + \Theta_\delta(t, s, \delta \gamma, \psi, \phi) - T_\epsilon(\delta \gamma, \phi) - M_\delta,
\]
and
\[
\Gamma_2(s, \phi) = V_1(t, \delta \gamma, \psi) - \Theta_\delta(t, \delta \gamma, s, \psi, \phi) + T_\epsilon(\delta \gamma, \phi) + M_\delta.
\]
Then we can get
\[
S(\Gamma_1)(t, \delta \gamma, \psi) = S(\Theta_\delta)(t, s, \delta \gamma, \psi, \phi) + S(\Phi_\delta)(t, s, \delta \gamma, \phi),
\]
and
\[
S(\Gamma_2)(s, \delta \gamma, \phi) = -S(\Theta_\delta)(t, s, \delta \gamma, \psi, \phi) + S(\Phi_\delta)(t, s, \delta \gamma, \phi).
\]
Therefore,
\[
S(\Gamma_1)(t, \delta \gamma, \psi) - S(\Gamma_2)(s, \delta \gamma, \phi) = S(\Theta_\delta)(t, s, \delta \gamma, \psi, \phi) + S(\Phi_\delta)(t, s, \delta \gamma, \phi). \tag{113}
\]
Recall that
\[
\Theta_\delta(t, s, \psi, \phi) = \frac{1}{\delta} \left[ F(\psi - \phi) + F(\psi_0 - \phi_0) + |t - s|^2 \right] + \gamma(G(\psi) + G(\phi)).
\]
Therefore, we can get
\[
\frac{1}{\delta} [S(\Theta_\delta) + S(\Phi_\delta)](t, s, \delta \gamma, \psi, \phi) \leq 0. \tag{115}
\]

Using Lemma 4.12 and Lemma 4.13, we deduce
\[
\frac{1}{\delta} [S(\Theta_\delta) + S(\Phi_\delta)](t, s, \delta \gamma, \psi, \phi) \leq 0.
\]


Thus, by virtue of (113), we have
\[
\limsup_{\delta, \epsilon \downarrow 0} \left[ S(\Gamma_1)(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - S(\Gamma_2)(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) \right] \leq 0. \quad (116)
\]

Combined with (112), the above inequality implies that
\[
\limsup_{\epsilon \downarrow 0, \delta \downarrow 0} \rho(V_1(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - V_2(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon})) \\
\leq \limsup_{\epsilon \downarrow 0, \delta \downarrow 0} \left\{ S(\Gamma_1)(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - S(\Gamma_2)(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) \right\} \\
+ \left[ L(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - [L(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon})] \right] \\
\leq \limsup_{\epsilon \downarrow 0, \delta \downarrow 0} \left\{ \left[ \left| L(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - L(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) \right| \right] \right\}. \quad (117)
\]

Using the Lipschitz continuity of \(L\) and Lemma 4.9, we can get that
\[
\limsup_{\delta, \epsilon \downarrow 0} |L(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - L(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon})| \\
\leq \limsup_{\delta, \epsilon \downarrow 0} \Lambda \left( |t_{\delta \gamma \epsilon} - s_{\delta \gamma \epsilon}| + \|\psi_{\delta \gamma \epsilon} - \phi_{\delta \gamma \epsilon}\|_2 \right) \\
= 0, \quad (118)
\]

Moreover, by virtue of (118) and (117), we have
\[
\limsup_{\epsilon \downarrow 0, \delta \downarrow 0} \rho(V_1(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - V_2(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon})) \leq 0. \quad (119)
\]

Since \((t_{\delta \gamma \epsilon}, s_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon})\) is the maximum of \(\Phi_{\delta \gamma} + T_\epsilon\) in \([0, T] \times [0, T] \times B \times B\), for all \((t, \psi) \in [0, T] \times B\), we have
\[
\Phi_{\delta \gamma}(t, \psi, \psi) + T_\epsilon(\psi, \psi) \leq \Phi_{\delta \gamma}(t_{\delta \gamma \epsilon}, s_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) + T_\epsilon(\psi_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}).
\]

Then we get
\[
\begin{align*}
V_1(t, \psi) - V_2(t, \psi) \\
\leq V_1(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - V_2(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) \\
- \frac{1}{2} \left[ \|\psi_{\delta \gamma \epsilon} - \phi_{\delta \gamma \epsilon}\|_2^2 + \|\phi_{\delta \gamma \epsilon} - \phi_{\delta \gamma \epsilon, 0}\|_2^2 + |t_{\delta \gamma \epsilon} - s_{\delta \gamma \epsilon}|^2 \right] \\
+ 2\gamma \exp(1 + \|\psi\|_2^2 + \|\psi_{\delta \gamma \epsilon, 0}\|_2^2) \\
+ \gamma \left[ \exp(1 + \|\psi_{\delta \gamma \epsilon, 0}\|_2^2 + \|\phi_{\delta \gamma \epsilon, 0}\|_2^2) + \exp(1 + \|\phi_{\delta \gamma \epsilon, 0}\|_2^2 + \|\phi_{\delta \gamma \epsilon, 0}\|_2^2) \right] \\
+ T_\epsilon(\psi_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) - T_\epsilon(\psi, \psi) \\
\leq V_1(t_{\delta \gamma \epsilon}, \psi_{\delta \gamma \epsilon}) - V_2(s_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) \\
+ 2\gamma \exp(1 + \|\psi\|_2^2 + \|\psi_{\delta \gamma \epsilon, 0}\|_2^2) + T_\epsilon(\psi_{\delta \gamma \epsilon}, \phi_{\delta \gamma \epsilon}) - T_\epsilon(\psi, \psi), \quad (120)
\end{align*}
\]
where the last inequality comes from the fact that $\delta > 0$ and $\gamma > 0$. By virtue of (111) and (119), when we take the $\lim \sup$ on (120) as $\delta, \epsilon$ and $\gamma$ go to zero, we can obtain
\[
V_1(t, \psi) - V_2(t, \psi) \leq \limsup_{\gamma \downarrow 0, \epsilon \downarrow 0, \delta \downarrow 0} \left( V_1(t \delta \gamma \epsilon, \psi \delta \gamma \epsilon) - V_2(s \delta \gamma \epsilon, \phi \delta \gamma \epsilon) 
+ 2\gamma \exp(1 + \|\psi\|^2 + \|\psi^0\|^2) + T_\epsilon(\psi \delta \gamma \epsilon, \phi \delta \gamma \epsilon) - T_\epsilon(\psi, \psi) \right) \leq 0.
\] (121)

Therefore, we have
\[
V_1(t, \psi) \leq V_2(t, \psi), \quad \forall (t, \psi) \in [0, T] \times B.
\] (122)

This completes the proof of Theorem 4.8. \qed

**Corollary 4.14** Let $V_1(t, \psi)$ and $V_2(t, \psi)$ be respectively a viscosity subsolution and a viscosity supersolution of (26) with at most a polynomial growth. Then we have
\[
V_1(t, \psi) \leq V_2(t, \psi), \quad \forall (t, \psi) \in [0, T] \times W^{1,2}((-r, 0); \mathbb{R}^n).
\] (123)

**Proof.** Let $(t, \psi) \in [0, T] \times W^{1,2}((-r, 0); \mathbb{R}^n)$. Then we can always find a closed ball $B$ of $W^{1,2}((-r, 0); \mathbb{R}^n)$ such that $\psi \in B$. Using Theorem 4.8, we know that $V_1(t, \phi) \leq V_2(t, \phi)$, for all $\phi \in B$. Therefore, $V_1(t, \psi) \leq V_2(t, \psi)$. This implies that
\[
V_1(t, \psi) \leq V_2(t, \psi), \quad \forall (t, \psi) \in [0, T] \times W^{1,2}((-r, 0); \mathbb{R}^n).
\]

This completes the proof. \qed

**Corollary 4.15** Let $V_1(t, \psi)$ and $V_2(t, \psi)$ be respectively a viscosity subsolution and a viscosity supersolution of (26) with at most a polynomial growth. Then, we have
\[
V_1(t, \psi) \leq V_2(t, \psi), \quad \forall (t, \psi) \in [0, T] \times C.
\] (124)

**Proof.** Since $W^{1,2}((-r, 0); \mathbb{R}^n)$ is dense in $C$, for any $\psi \in C$ we can always find a sequence $(\psi_k)_k$ with $\psi_k \in W^{1,2}((-r, 0); \mathbb{R}^n)$ such that
\[
\lim_{k \to \infty} \|\psi_k - \psi\| \to 0.
\]

Using Corollary 4.14 we know that $V_1(t, \psi_k) \leq V_2(t, \psi_k)$ for all $t \in [0, T]$. Therefore, for all $t \in [0, T]$, using the continuity of $V_1$ and $V_2$, we can get
\[
V_1(t, \psi) = \lim_{k \to \infty} V_1(t, \psi_k) \leq \lim_{k \to \infty} V_2(t, \psi_k) = V_2(t, \psi).
\] (125)
This completes the proof. □

Now we give the main result of this paper:

**Theorem 4.16 (Uniqueness)** The value function \( V : [0,T] \times C \to R \) defined by (11) is the unique viscosity solution of the HJBVI (26).

**Proof.** Since the value function \( V : [0,T] \times C \to R \) is a viscosity solution (and hence is both a subsolution and a supersolution) of the HJBVI (26), the uniqueness result of the viscosity solution follows immediately from the above comparison principle. □

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**References**


