MA (ST) 413 Midterm Exam 2
Solutions

1. (10 points) Use the formula for pgf, we have:

$$\frac{dP^M(z)}{dz} = \frac{1-p_0^M}{1-p_0}P'(z).$$

Therefore, we have

$$\begin{aligned} Pr(N=1) &= \left.\frac{dP^M(z)}{dz}\right|_{z=0} = \left.\frac{1-p_0^M}{1-p_0}P'(z)\right|_{z=0} \\ &= \frac{(1-p_0^M)p_1}{1-p_0} = \frac{(1-0.5)\lambda e^{-\lambda}}{1-e^{-\lambda}} \\ &= 0.1565. \end{aligned}$$

2. By assumptions, we have $\alpha = 0.8, d = 200$ and $\alpha(u-d) = 3000$. So we can get $u = 4000$.

(a) (10 points) From the formula sheet, we have

$$f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \quad F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha,$$

$$E[X \wedge x] = \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1.$$

In addition, we know that

$$Y^L = 0.8[X \wedge 4000 - X \wedge 250].$$

Therefore, we can get

$$\begin{aligned} E[Y^L] &= 0.8(E[X \wedge 4000] - -E[X \wedge 250]) \\ &= 0.8\left(\frac{1000}{3-1}\left[1-\left(\frac{1000}{1000+4000}\right)^{3-1}\right]\right. \\ &\qquad \left. - \frac{1000}{3-1}\left[1-\left(\frac{1000}{1000+250}\right)^{3-1}\right]\right) \\ &= 0.8(480-180) \\ &= 240. \end{aligned}$$

(b) (10 points) Using the formula in (a), we have

$$E[X] = E[X \wedge \infty] = \frac{\theta}{\alpha - 1} = \frac{1000}{3-1} = 500.$$

Therefore, we can get

$$\begin{aligned} \frac{E[X] - E[Y^L]}{E[X]} &= \frac{500 - 240}{500} \\ &= 52\%. \end{aligned}$$

(c) (10 points) On the other hand, we know that $E[Y^P] = \frac{E[Y^L]}{1-F_X(d)}$. So we have

$$\begin{aligned} E[Y^P] &= \frac{E[Y^L]}{1 - F_X(d)} = \frac{240}{1 - \left(1 - (\frac{1000}{1000+250})^3\right)} \\ &= \frac{240}{0.512} = 468.75. \end{aligned}$$

(d) (10 points) The distribution of N^P is also a binomial distribution and we only need to calculate the new parameter q^*:

$$\begin{aligned} v &= 1 - Pr(X \leq d) = 1 - \left(1 - \left(\frac{1000}{1000 + 250}\right)^3\right) \\ &= .8^3 = .512, \\ q^* &= vq = 0.512 \cdot 0.01 = 0.00512. \end{aligned}$$

So N^P follows a binomial distribution with $m = 1000, q^* = 0.00512$.

3. (a) (10 points) Using convolution process, we can get

k	$f_{X_1}(k)$	$f_{X_2}(k)$	$f_Y(k)$
1	0.5	0.5	0
2	0.4	0.4	0.25
3	0.1	0.1	0.40
4	0	0	0.26
5	0	0	0.08
6	0	0	0.01

(b) (20 points) Again, using the convolution process, we can get

k	$f^{*(0)}$	$f^{*(1)}(k)$	$f^{*(2)}(k)$	$f^{*(3)}(k)$	$f^{*(4)}(k)$	$f_S(k)$
0	1	0	0	0	0	e^{-2}
1	0	0.5	0	0	0	e^{-2}
2	0	0.4	0.25	0	0	$1.3e^{-2}$
3	0	0.1	0.40	0.125	0	$1.167e^{-2}$
4	0	0	0.26	0.3	0.0625	$0.9617e^{-2}$
n	0	1	2	3	4	
$Pr(N=n)$	e^{-2}	$2e^{-2}$	$2e^{-2}$	$\frac{4}{3}e^{-2}$	$\frac{2}{3}e^{-2}$	

Therefore, we have

$$Pr(S=4) = 0.9617e^{-2} = 0.1302.$$

4. (a) (10 points)

$$\begin{aligned} E[S] &= E[N]E[X] = r\beta \cdot 2 = 5 \cdot 4 \cdot 2 = 40, \\ Var(S) &= E[N]Var(X) + (E[X])^2 Var(N) \\ &= r\beta\theta^2 + \theta^2 r\beta(\beta+1) \\ &= 5 \cdot 4 \cdot 2^2 + \cdot 2^2 \cdot 5 \cdot 4 \cdot (4+1) \\ &= 480. \end{aligned}$$

(b) (10 points) Using normal approximation, we should have

$$\frac{(1+\gamma)E[S] - E[S]}{\sqrt{Var(S)}} = \pi_{0.95} = 1.645.$$

So we have

$$\gamma = \pi_{0.95}\frac{\sqrt{Var(S)}}{E[S]} = 1.645\frac{\sqrt{480}}{40} = 0.90.$$