MA (ST)413 Midterm Exam 1
Solutions

1. (20 points)

   (a) (10 points)

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{50} - \frac{x^2}{100^2}, & \text{if } 0 \le x \le 100, \\ 1, & \text{if } x > 100. \end{cases}$$

$$S_X(x) = 1 - F_X(x) = \begin{cases} 1, & \text{if } x < 0, \\ 1 - \frac{x}{50} + \frac{x^2}{100^2}, & \text{if } 0 \le x \le 100, \\ 0, & \text{if } x > 100. \end{cases}$$

$$h_X(x) = \frac{f_X(x)}{S_X(x)} = \begin{cases} 0, & \text{if } x < 0, \\ \frac{200-2x}{10000-200x+x^2}, & \text{if } 0 \le x \le 100. \end{cases}$$

   (b) (10 points) By definition and using integration by parts, we can get

$$EX = \int_0^{100} x f_X(x)dx = \int_0^{100} x\frac{1}{50}\left(1 - \frac{x}{100}\right)dx = \frac{100}{3} = 33.33,$$

$$E[X^2] = \int_0^{100} x^2 f_X(x)dx = \int_0^{100} x^2\frac{1}{50}\left(1 - \frac{x}{100}\right)dx = \frac{10000}{6} = 1666.67,$$

$$Var[X] = E[X^2] - [EX]^2 = \frac{10000}{18} = 555.56.$$

2. (20 points)

   (a) (10 points) Let $X_1 = cX$. Then the pdf of X_1 is given by

$$\begin{aligned} f_{X_1}(x) &= \frac{1}{c} f_X(x/c) \\ &= \frac{1}{c} \cdot \frac{\alpha_1 \theta_1^{\alpha_1}}{(\frac{x}{c} + \theta_1)^{\alpha_1+1}} \\ &= \frac{\alpha_1 (c\theta_1)^{\alpha_1}}{(x + c\theta_1)^{\alpha_1+1}}. \end{aligned}$$

   This is the pdf for a Pareto distribution with parameters $(\alpha_1, c\theta_1)$. So Pareto distribution is a scale distribution with a scale parameter θ_1.
   Let Y follows a Gamma distribution with parameters (α_2, θ_2) and let $Y_1 = cY$. Then the pdf of Y_1 is given by

$$\begin{aligned} f_{Y_1}(x) &= \frac{1}{c} f_Y(x/c) \\ &= \frac{1}{c} \cdot \frac{\left(\frac{x}{c}\right)^{\alpha_2 - 1} e^{-\frac{x}{c\theta_2}}}{\theta_2^{\alpha_2}\Gamma(\alpha_2)} \\ &= \frac{x^{\alpha_2-1} e^{-\frac{x}{c\theta_2}}}{(c\theta_2)^{\alpha_2}\Gamma(\alpha_2)}. \end{aligned}$$

This is the pdf of the Gamma distribution with parameters $(\alpha_2, c\theta_2)$. So Gamma distribution is a scale distribution with a scale parameter θ_2.

(b) (10 points) We consider the ratio of two pdfs. Actually, as $x \to \infty$, we have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1 \theta_1^{\alpha_1} \theta_2^{\alpha_2} \Gamma(\alpha_2)}{(x+\theta_1)^{\alpha_1+1} x^{\alpha_2-1} e^{-\frac{x}{\theta_2}}} \sim \frac{e^{\frac{x}{\theta_2}}}{(x+\theta_1)^{\alpha_1+1} x^{\alpha_2-1}}.$$

Since the growth rate of the exponential function is larger than polynomials, we can get that

$$\lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)} = \infty.$$

Therefore, Pareto distribution has a heavier tail.

3. (10 points)

$$\begin{aligned} f_Y(y) &= \frac{1}{1+r} f_X\left(\frac{y}{1+r}\right) \\ &= \frac{1}{1.04}\left(2e^{-\frac{4y}{1.04}} + 3e^{-\frac{6y}{1.04}}\right). \end{aligned}$$

4. (10 points) Since $X \sim N(1, 4^2)$, we can get

$$\frac{X-1}{2} \sim N(0, 2^2).$$

Therefore, $\tilde{Y} \equiv e^{\frac{X-1}{2}}$ follows a log-normal distribution with parameters 0 and 2^2. In other words, the pdf of $\tilde{Y} \equiv e^{\frac{X-1}{2}}$ is

$$f_{\tilde{Y}}(x) = \frac{1}{2x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2\cdot 2^2}}.$$

Since $Y = 2\left[e^{\frac{X-1}{2}}\right] = 2\tilde{Y}$, the pdf of Y is given by

$$f_Y(y) = \frac{1}{2} f_{\tilde{Y}}(y/2) = \frac{1}{2y\sqrt{2\pi}} e^{-\frac{(\ln y - \ln 2)^2}{2\cdot 2^2}}.$$

5. (20 points) First we calculate $E[\Theta]$.

$$\begin{aligned} E[\Theta] &= \int_0^\infty \theta \cdot \frac{\theta^{\alpha-1} e^{-\frac{\theta}{\beta}}}{\beta^\alpha \Gamma(\alpha)} d\theta \\ &= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{\theta^{(\alpha+1)-1} e^{-\frac{\theta}{\beta}}}{\beta^{\alpha+1}\Gamma(\alpha+1)} d\theta \\ &= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \\ &= \alpha\beta \end{aligned}$$

Then the unconditional expectation of N is

$$E[N] = E[E[N|\Theta]] = E[\lambda\Theta] = \lambda E[\Theta] = \lambda\alpha\beta.$$

6. (20 points) Let N_1, N_2 denote the number of accidents for a driver from Class I, II, respectively. Then, we can get

$$\begin{aligned} Pr(N_1 = 0) &= \int_2^5 e^{-\lambda} \frac{1}{5-2} d\lambda = \frac{1}{3}\left[e^{-2} - e^{-5}\right] = 0.0429. \\ Pr(N_2 = 0) &= \int_1^3 e^{-\lambda} \frac{1}{3-1} d\lambda = \frac{1}{2}\left[e^{-1} - e^{-3}\right] = 0.1590. \end{aligned}$$

Let N denote the number of accidents for a driver from this population. Then N is a $(0.35, 0.65)$ mixture of N_1, N_2. Therefore,

$$Pr(N = 0) = 0.35 \cdot Pr(N_1 = 0) + 0.65 \cdot Pr(N_2 = 0) = 0.1184.$$

Thus, $Pr(N \geq 1) = 1 - Pr(N = 0) = 1 - 0.1184 = 0.8816.$