MA (ST) 413 Assignment 10
Solutions

1. (page 276, Problem 10.1) The premium is collected at the beginning of each year and it can earn interest. Using the given data, we have

$$U_0^* = u = 2, \quad C_1 = 0.1(U_0^* + P) = 0.1(2+6) = 0.8,$$

$$W_1 = P_1 + C_1 - S_1 = \begin{cases} 6.8 & (0.4), \\ 1.8 & (0.3), \\ -3.2 & (0.15), \\ -8.2 & (0.1), \\ -13.2 & (0.05), \end{cases}$$

where the values in ( ) stand for the corresponding probabilities. Since $U_0 > 0$, we have that $W_1^* = W_1$. Thus, we can get

$$U_1^* = U_0^* + W_1^* = U_0^* + W_1 = \begin{cases} 8.8 & (0.4), \\ 3.8 & (0.3), \\ -1.2 & (0.15), \\ -6.2 & (0.1), \\ -11.2 & (0.05). \end{cases}$$

For $U_1^* < 0$, it will lead to ruin. Then we can rewrite U_1^* as

$$U_1^* = U_0^* + W_1^* = U_0^* + W_1 = \begin{cases} 8.8 & (0.4), \\ 3.8 & (0.3), \\ \textit{ruin} & (0.3). \end{cases}$$

In other words, after the period 1, the total probability of ruin is 0.3.

For $U_1^* = 8.8$, we have

$$C_2 = 0.1(U_1^* + P_2) = 1.48, \quad W_2 = P_2 + C_2 - S_2 = \begin{cases} 7.48 & (0.4), \\ 2.48 & (0.3), \\ -2.52 & (0.15), \\ -7.52 & (0.1), \\ -12.52 & (0.05). \end{cases}$$

Again, since $U_1^* > 0$ we have that $W_2^* = W_2$. So we can get

$$U_2^*(U_1^* = 8.8) = \begin{cases} 16.28 & (0.4), \\ 11.28 & (0.3), \\ 6.28 & (0.15), \\ 1.28 & (0.1), \\ -2.72 & (0.05). \end{cases}$$

Similarly, for $U_1^* = 3.8$, we can get

$$\begin{aligned} C_2 &= 0.1(U_1^* + P_2) = 0.98, \\ U_2^*(U_1^* = 3.8) &= \begin{cases} 10.78 & (0.4), \\ 5.78 & (0.3), \\ 0.78 & (0.15), \\ -4.22 & (0.1), \\ -9.22 & (0.05). \end{cases} \end{aligned}$$

Therefore, at time 2, we have

$$U_2^* = \begin{cases} 16.28 & (0.4 \cdot 0.4 = 0.16), \\ 11.28 & (0.4 \cdot 0.3 = 0.12), \\ 6.28 & (0.4 \cdot 0.15 = .06), \\ 1.28 & (0.4 \cdot 0.1 = 0.04), \\ -2.72 & (0.4 \cdot 0.05 = 0.02); \\ 10.78 & (0.3 \cdot 0.4 = 0.12), \\ 5.78 & (0.3 \cdot 0.3 = 0.09), \\ 0.78 & (0.3 \cdot 0.15 = 0.045), \\ -4.22 & (0.3 \cdot 0.1 = 0.03), \\ -9.22 & (0.3 \cdot 0.05 = 0.015), \\ \text{ruin at time 1} & (0.3). \end{cases}$$

Put all negative values together, and we can get

$$U_2^* = \begin{cases} 16.28 & (0.16), \\ 11.28 & (0.12), \\ 10.78 & (0.12), \\ 6.28 & (0.06), \\ 5.78 & (0.09), \\ 1.28 & (0.04), \\ 0.78 & (0.045), \\ \text{ruin at time 2} & (0.365). \end{cases}$$

Continue this process, and at time 3, priori to claims, the surplus is given by

$$\begin{cases} 24.508 & (0.16), \\ 19.008 & (0.12), \\ 18.458 & (0.12), \\ 13.508 & (0.06), \\ 12.958 & (0.09), \\ 8.008 & (0.04), \\ 7.7458 & (0.045). \end{cases}$$

The claim size can be 0,5,10, 15, 20. For 24.505, no claims will make U_3^* negative. For 19.008, only claim of size 20 will make $U_3^* < 0$. The probability contributed to ruin is $0.12(0.05) = 0.006$. Similarly, we can get the total contribution to probability of ruin:

$$0.16(0)+0.12(0.005)+0.12(0.05)+0.06(0.15)+0.09(0.15)+0.04(0.3)+0.045(0.3) = 0.06.$$

So the overall probability of ruin is $0.365 + 0.06 = 0.425$. That is

$$\tilde{\psi}(2,3) = 0.425.$$

2. (page 276, Problem 10.2) This problem is similar to (10.1). The premium is collected at the beginning of each year and it can earn interest. Using the given data, we have

$$U_0^* = u = 1, \quad C_1 = 0.1(U_0^* + P_1) = 0.1(1+2) = 0.3,$$
$$W_1 = P_1 + C_1 - S_1 = \begin{cases} 2.3 & (0.6), \\ 0.3 & (0.3), \\ -3.7 & (0.1). \end{cases}$$

where the values in ( ) stand for the corresponding probabilities. Since $U_0 > 0$, we have that $W_1^* = W_1$. Thus, we can get ($U_1^* < 0$ leads to ruin)

$$U_1^* = U_0^* + W_1^* = U_0^* + W_1 = \begin{cases} 3.3 & (0.6), \\ 1.3 & (0.3), \\ ruin & (0.1). \end{cases}$$

After the period 1, the total probability of ruin is 0.1.

For $U_1^* = 3.3$, we have

$$C_2 = 0.1(U_1^* + P_2) = 0.53, \quad W_2 = P_2 + C_2 - S_2 = \begin{cases} 2.53 & (0.6), \\ 0.53 & (0.3), \\ -4.47 & (0.1). \end{cases}$$

Again, since $U_1^* > 0$ we have that $W_2^* = W_2$. So we can get

$$U_2^*(U_1^* = 3.3) = \begin{cases} 5.83 & (0.6), \\ 3.83 & (0.3), \\ -0.17 & (0.1). \end{cases}$$

Similarly, for $U_1^* = 1.3$, we can get

$$\begin{aligned} C_2 &= 0.1(U_1^* + P_2) = 0.33, \\ U_2^*(U_1^* = 1.3) &= \begin{cases} 3.63 & (0.6), \\ 1.63 & (0.3), \\ -2.37 & (0.1). \end{cases} \end{aligned}$$

Therefore, at time 2, we have

$$U_2^* = \begin{cases} 5.83 & (0.6 \cdot 0.6 = 0.36), \\ 3.83 & (0.6 \cdot 0.3 = 0.18), \\ -0.17 & (0.6 \cdot 0.1 = 0.06), \\ 3.63 & (0.3 \cdot 0.6 = 0.18), \\ 1.63 & (0.3 \cdot 0.3 = 0.09), \\ -2.37 & (0.3 \cdot 0.1 = 0.03), \\ \text{ruin at time 1} & (0.1). \end{cases}$$

Put all negative values together, and we can get

$$U_2^* = \begin{cases} 5.83 & (0.36), \\ 3.83 & (0.18), \\ 3.63 & (0.18), \\ 1.63 & (0.09), \\ \text{ruin at time 2} & (0.19). \end{cases}$$

Continue this process, and at time 3, priori to claims, the surplus is given by

$$\begin{cases} 8.613 & (0.36), \\ 6.413 & (0.18), \\ 6.193 & (0.18), \\ 3.993 & (0.09). \end{cases}$$

The claim size can be 0, 2, 6. So only for the last case and the claim size of 6, there will be a ruin. The probability contributed to ruin is $0.09(0.1) = 0.009$. So the overall probability of ruin is $0.19 + 0.009 = 0.199$. That is

$$\tilde{\psi}(1,3) = 0.199.$$

3. Let $W_i \equiv P_i + C_i - S_i$ be the net income from the $i - th$ period. Then W_i has the following distribution:

$$Pr(W_i = 4) = 0.7, \quad Pr(W_i = -4) = 0.3.$$

In other words, W_i can only take values 4 and -4. Since $u = 5$, the surplus just before the ruin must be $5 - 4 = 1$, therefore, the value of $U(\tilde{T})$ must be $1 - 4 = -3$.