MA (ST) 413 Assignment 9
Solutions

1. (page 241, Problem 9.64) It is easy to know that

$$v = 1 - F_X(d) = e^{-\mu d}.$$

Then from formula (9.31), we have

$$P_{N^P}(x) = B(\theta(1 - v + vz - 1)] = B[\theta v(z-1)] = B[\theta e^{-\mu d}(z-1)].$$

On the other hand, using $Y^P = \alpha(X-d)|X > d$, we can get

$$\begin{aligned} F_{Y^P}(y) &= Pr(Y^P \le y) = \frac{Pr(\alpha(X-d) \le y, X > d)}{Pr(X > d)} \\ &= \frac{Pr(d < X \le \frac{y}{\alpha} + d)}{e^{-\mu d}} \\ &= 1 - e^{-\frac{\mu}{\alpha} y}. \end{aligned}$$

This completes the proof.

2. (page 259, Problem 9.75) For a r.v. Y following an exponential distribution with mean θ, we have

$$E[Y] = \theta, \quad E[Y^2] = 2\theta^2, \quad Var(Y) = \theta^2.$$

Therefore, for an individual insured with claim probability q, the total loss amount X satisfies:

$$E[X] = qE[Y] = \theta q, \quad Var(X) = qVar(Y) + q(1-q)(E[Y])^2 = q(2-q)\theta^2.$$

Therefore, we can get

$$\begin{aligned} E[S] &= \sum_i E[X_i] \\ &= 400(0.03)(5) + 300(0.07)(3) + 200(0.10)2 \\ &= 163; \\ Var(S) &= \sum_i Var(X_i) \\ &= 400(0.03)(1.97)(25) + 300(0.07)(1.93)(9) \\ &\quad + 200(0.10)(1.90)(4) \\ &= 1107.77. \end{aligned}$$

Let P be premium. Then we have

$$\begin{aligned} & Pr(S \le P) = 0.05 \\ \Rightarrow\ & Pr\left(\frac{S - E[S]}{\sqrt{Var(S)}} \le \frac{P - E[S]}{\sqrt{Var(S)}}\right) = 0.05 \\ \Rightarrow\ & P = E[S] + 1.645\sqrt{Var(S)} = 217.75. \end{aligned}$$

3. (page 259, Problem 9.76) Let X_1, X_2, X_3 be the claim amount for office visit, surgery and other services respectively. Let X be the total claim amount for each individual. Then we have

$$X = X_1 + X_2 + X_3.$$

Using the given data and equations (9.43), (9.44), we can get

$$\begin{aligned} E[X_1] &= 0.7(160) = 112, \\ Var(X_1) &= 0.7(4900) + 0.7(1-0.7)(160)^2 = 8806, \\ E[X_2] &= 0.2(600) = 120, \\ Var(X_2) &= 0.2(20000) + 0.2(1-0.2)(600)^2 = 61600, \\ E[X_3] &= 0.5(240) = 120, \\ Var(X_3) &= 0.5(8100) + 0.5(1-0.5)(240)^2 = 18450. \end{aligned}$$

Since X_1, X_2, X_3 are independent, we have

$$\begin{aligned} E[X] &= E[X_1] + E[X_2] + E[X_3] = 112 + 120 + 120 = 352, \\ Var(X) &= Var(X_1) + Var(X_2) + Var(X_3) = 88856. \end{aligned}$$

Assume there are n members. Then we have

$$E[S] = nE[X] = 352n, \quad Var(S) = nVar(X) = 88856n.$$

The goal is

$$Pr(S > 1.15E[S]) < 0.1,$$

which is equivalent to

$$Pr\left(\frac{S - E[S]}{\sqrt{Var(S)}} \le \frac{1.15E[S] - E[S]}{\sqrt{Var(S)}}\right) \ge 0.9.$$

The 90th percentile of the standard normal distribution is 1.28155. Therefore, we have

$$\frac{1.15E[S] - E[S]}{\sqrt{Var(S)}} > 1.28155.$$

That is

$$.15(352n) > 1.28155\sqrt{88856n}.$$

Solve for n to get $n > 52.35$. So 53 members are needed.

4. (page 259, Problem 9.78) Set $\lambda_i = q_i$. Then we get

$$\lambda = \sum \lambda_i = 500(0.01) + 500(0.02) = 15.$$

In addition, the corresponding individual claim distribution is given by

$$f_Y(y) = \sum_i \frac{\lambda_i}{\lambda} f_{Y_i}(y).$$

Then we can get

$$f_Y(x) = \frac{500(0.01)}{15} = \frac{1}{3},$$
$$f_Y(2x) = \frac{500(0.02)}{15} = \frac{2}{3}.$$

So we have

$$E[X^2] = 3x^2, \quad Var(S) = \lambda E[X^2] = 45x^2.$$

Set $Var(S) = 4500$ and we can get $X = 10$.