MA (ST) 413 Assignment 8 Solutions

1. (Problem 9.23) Define $S_n = X_1 + X_2 + \cdots + X_n$. Since $X_1, X_2, \cdots$ are i.i.d. r.v.'s and they follow a distribution $N(100, 9)$, we have

$$S_n \sim N(100n, 9n).$$

Then we can get

$$Pr(S_n > 100) = Pr\left(\frac{S_n - 100n}{\sqrt{9n}} > \frac{100 - 100n}{\sqrt{9n}}\right) = 1 - \Phi\left(\frac{100 - 100n}{\sqrt{9n}}\right),$$

where Φ is the cdf of the standard normal distribution. Now we can get

$$\begin{aligned}
Pr(S > 100) &= \sum_{i=0}^{3} Pr(N = n) Pr(S_n > 100) \\
&= \sum_{i=0}^{3} Pr(N = n) \left[1 - \Phi\left(\frac{100 - 100n}{\sqrt{9n}}\right)\right] \\
&= 0.5 \cdot 0 + 0.2(1 - \Phi(0)) + 0.2\left[1 - \Phi\left(\frac{-100}{\sqrt{18}}\right)\right] \\
&\quad + 0.1\left[1 - \Phi\left(\frac{-200}{\sqrt{27}}\right)\right] \\
&= 0.2 \cdot 0.5 + 0.2 \cdot 1 + 0.1 \cdot 1 \\
&= 0.4.
\end{aligned}$$

2. (Problem 9.24) Using the convolution process, we can get

k	$f^{*0}(k)$	$f_{X_1}(k)$	$f_{X_1+X_2}(k)$	$f_S(k)$
0	1	0	0	0.0625
2000	0	0.4	0	0.1
3000	0	0.6	0	0.15
4000	0	0	0.16	0.06
n	0	1	2	
$f(n)$	1/16	1/4	3/8	

Then we have

$$\begin{aligned}
&Pr(S \geq 5000) \\
= \;& 1 - Pr(S = 0) - Pr(S = 2000) - Pr(S = 3000) - Pr(S = 4000) \\
= \;& 0.6275.
\end{aligned}$$

Therefore, the total cost is

$$1472 + 2000(0.1) + 3000(0.15) + 4000(0.06) + 5000(0.6275) = 5499.5.$$

3. (Problem 9.26) Using the pdf of S, we can get

$$\begin{aligned} E[S] &= \int_1^\infty x3x^{-4}dx = \frac{3}{2}, \\ E[S^2] &= \int_1^\infty x^2 3x^{-4}dx = 3, \\ Var(S) &= E[S^2] - (E[S])^2 = \frac{3}{4}. \end{aligned}$$

Then we have

$$\begin{aligned} 0.9 &= Pr\left(S \le (1+\theta)\frac{3}{2}\right) \\ &= \int_1^{(1+\theta)1.5} 3x^{-4}dx \\ &= 1 - [1.5(1+\theta)]^{-3}. \end{aligned}$$

So we can get that $\theta = 0.4363$. On the other hand, we have

$$\begin{aligned} 0.9 &= Pr(S \le 1.5 + \lambda\sqrt{0.75}) \\ &= \int_1^{1.5+\lambda\sqrt{0.75}} 3x^{-4}dx \\ &= 1 - [1.5 + \lambda\sqrt{0.75}]^{-3}, \end{aligned}$$

and we can get $\lambda = 0.7557$.

4. (Problem 9.27) The answer is

$$0.8R_{100} + 0.1R_{1100} + 0.1R_{2100},$$

where $0.8R_{100}$ pays 80% of all in excess of 100, $0.1R_{1100}$ pays an additional 10% in excess of 1100, and $0.1R_{2100}$ pays an additional 10% in excess of 2100.

5. (Problem 9.31) The total claim follows a compound Poisson with $\lambda = 5$ and severity distribution

$$\begin{aligned} f_X(x) &= 0.4f_1(x) + 0.6f_2(x) \\ &= \begin{cases} 0.4(0.001) + 0.6(0.005) = 0.0034, & 0 < x \le 200; \\ 0.4(0.001) + 0.6(0) = 0.0004, & 200 < x \le 1000. \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} E[(X-100)_+] &= \int_{100}^\infty (x-100)f_X(x)dx \\ &= \int_{100}^{200} (x-100)(0.0034)dx + \int_{200}^{1000} (x-100)(0.0004)dx \\ &= 177. \end{aligned}$$

6. (Problem 9.38) Let $\hat{X}$ be the r.v. with probability function $g_X(x), x = 1, 2, \cdots$. Then we have

$$\begin{aligned} M_{\hat{X}}(t) &= E[e^{t\hat{X}}] \\ &= \sum_{x=1}^{\infty} e^{tx} g_X(x) \\ &= \frac{1}{1 - f_X(0)} \sum_{x=1}^{\infty} e^{tx} f_X(x) \\ &= \frac{M_X(t) - f_X(0)}{1 - f_X(0)}. \end{aligned}$$

For compound negative binomial with f_X, we have

$$M_S(t) = P_N(M_X(t)) = [1 - \beta(M_X(t) - 1)]^{-2}.$$

Let $\hat{N}$ be the new frequency distribution with severity distribution $g_X(x)$ which keeps the same compound distribution. Then we should have

$$\begin{aligned} M_S(t) &= [1 - \beta(M_X(t) - 1)]^{-2} \\ &= P_{\hat{N}}(M_{\hat{X}}(t)). \end{aligned}$$

It is easy to verify that if $\hat{N}$ follows a negative binomial with $r = 2, \hat{\beta} = 1 - f_X(0)$, then we have

$$\begin{aligned} M_S(t) &= P_{\hat{N}}(M_{\hat{X}}(t)) \\ &= [1 - \hat{\beta}(M_{\hat{X}}(t) - 1)]^{-2} \\ &= \left[1 - \beta(1 - f_X(0)) \left(\frac{M_X(t) - f_X(0)}{1 - f_X(0)} - 1 \right) \right]^{-2} \\ &= \left[1 - \beta(1 - f_X(0)) \left(\frac{M_X(t) - 1)}{1 - f_X(0)} \right) \right]^{-2} \\ &= [1 - \beta\left(M_X(t) - 1\right)]^{-2}. \end{aligned}$$

Therefore, the new parameter for the frequency distribution is $\hat{\beta} = \beta(1 - f_X(0))$ and $r = 2$.

7. (Problem 9.39) Let $S_n = X_1 + X_2 + \cdots + X_n$, and let $f_n(x)$ be the pdf of S_n. The mgf of X_i is

$$M_X(t) = \frac{1}{1 - \theta t}.$$

Then the mgf of S_n is

$$M_{S_n}(t) = [M_X(t)]^n = \frac{1}{(1 - \theta t)^n}.$$

Therefore, S_n follows a Gamma distribution with parameters n, θ. That is

$$f_n(x) = \frac{x^{n-1} e^{-\frac{x}{\theta}}}{\theta^n \Gamma(n)} = \frac{x^{n-1} e^{-\frac{x}{\theta}}}{\theta^n (n-1)!}.$$

(a.) Using the results we obtained above, we can get

$$\begin{aligned} f_S(x) &= \sum_{n=1}^{\infty} Pr(N=n) f_n(x) \\ &= \sum_{n=1}^{\infty} \frac{\beta^n}{n(1+\beta)^n \ln(1+\beta)} \cdot \frac{x^{n-1} e^{-\frac{x}{\theta}}}{\theta^n (n-1)!} \\ &= \frac{1}{\ln(1+\beta)} \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{\beta}{\theta(1+\beta)}\right]^n x^{n-1} e^{-\frac{x}{\theta}}. \end{aligned}$$

(b.)

$$\begin{aligned} f_S(x) &= \frac{1}{\ln(1+\beta)} \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{\beta}{\theta(1+\beta)}\right]^n x^{n-1} e^{-\frac{x}{\theta}} \\ &= \frac{e^{-\frac{x}{\theta}}}{\ln(1+\beta)} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{x} \left[\frac{\beta x}{\theta(1+\beta)}\right]^n \\ &= \frac{e^{-\frac{x}{\theta}}}{x \ln(1+\beta)} \left[e^{\frac{\beta x}{\theta(1+\beta)}} - 1\right] \\ &= \frac{e^{-\frac{x}{\theta(1+\beta)}} - e^{-\frac{x}{\theta}}}{x \ln(1+\beta)}. \end{aligned}$$