MA (ST) 413 Assignment 4
Solutions

1.

$$
\begin{aligned}
E[e^{3X}] &= \int_{-\infty}^{\infty} e^{3x} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= e^{3\mu+\frac{9}{2}\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-(\mu+3\sigma^2)}{\sigma}\right)^2} dx \\
&= e^{3\mu+\frac{9}{2}\sigma^2}. \\
E[e^{4X}] &= \int_{-\infty}^{\infty} e^{4x} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= e^{4\mu+8\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-(\mu+4\sigma^2)}{\sigma}\right)^2} dx \\
&= e^{4\mu+8\sigma^2}. \\
M_X(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= e^{t\mu+\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-(\mu+t\sigma^2)}{\sigma}\right)^2} dx \\
&= e^{t\mu+\frac{t^2\sigma^2}{2}}.
\end{aligned}
$$

2. Let X_1, X_2, X_3 denote the number of accidents for a driver from Class I, II, III, respectively. Then, we can get

$$
\begin{aligned}
Pr(X_1 = 0) &= \int_{0.2}^{1.0} e^{-\lambda} \frac{1}{1.0-0.2} d\lambda = \frac{1}{0.8}\left[e^{-0.2} - e^{-1.0}\right] = 0.5636. \\
Pr(X_2 = 0) &= \int_{0.4}^{2.0} e^{-\lambda} \frac{1}{2.0-0.4} d\lambda = \frac{1}{1.6}\left[e^{-0.4} - e^{-2.0}\right] = 0.3343. \\
Pr(X_3 = 0) &= \int_{0.6}^{3.0} e^{-\lambda} \frac{1}{3.0-0.6} d\lambda = \frac{1}{2.4}\left[e^{-0.6} - e^{-3.0}\right] = 0.2079.
\end{aligned}
$$

Let Y denote the number of accidents for a driver from this population. Then Y is a $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ mixture of X_1, X_2, X_3. Therefore,

$$
Pr(Y = 0) = \frac{Pr(X_1 = 0) + Pr(X_2 = 0) + Pr(X_3 = 0)}{3} = 0.3686.
$$

3. The claims under the new policies follows a Poisson with a parameter $\lambda_{new} = 50 \cdot 0.92 = 46$.

4. Let N_i be the number of poisoned wine glasses used by the King in the $i-th$ day. Then N_i follows a Poisson distribution with $\lambda_i = 2 \cdot 0.02 = 0.04$. Let

$N = N_1 + N_2 + \cdots + N_{50}$, then N is the total number of poisoned glassed used by the King during the first 50 days. Apparently, N follows a Poisson distribution with $\lambda = 50 \cdot \lambda_1 = 50 \cdot 0.04 = 2$. The probability we are looking for is

$$Pr(N = 0) = e^{-\lambda} = e^{-2} = 0.1353.$$

5. Let X be a r.v. standing for the number of claims. Since X follows the geometric distribution with $\beta = 2$, we have

$$Pr(X = 0) = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

Therefore,

$$Pr(X \geq 1) = 1 - Pr(X = 0) = \frac{2}{3}.$$

6. **Method 1:** According to the definition of Binomial distribution, we have

$$\begin{aligned} E[2^N] &= \sum_{k=0}^{20} 2^k \cdot Pr(N = k) \\ &= \sum_{k=0}^{20} 2^k \cdot \binom{20}{k} 0.2^k \cdot (1 - 0.2)^{20-k} \\ &= \sum_{k=0}^{20} \binom{20}{k} (2 \cdot 0.2)^k \cdot 0.8^{20-k} \\ &= (0.4 + 0.8)^{20} \\ &= 38.34. \end{aligned}$$

Method 2: As we know, the pdf of N is

$$P_N(z) \equiv E[z^N] = [1 + 0.2(z - 1)]^{20}.$$

Plug in $z = 2$, and we can get

$$E[2^N] = P_N(2) = [1 + 0.2(2 - 1)]^{20} = 1.2^{20} = 38.34.$$

7. (Problem 6.1) By the definitions of the pgf and mgf, we have

$$P_N(z) = \sum_{k=0}^{\infty} p_k z^k = \sum_{k=0}^{\infty} p_k e^{k \ln z} = M_N(\ln z).$$

Therefore, we can get

$$\begin{aligned} P'_N(z) &= z^{-1} M'_N(\ln z), \\ P'_N(1) &= M'(0) = E[N], \\ P''_N(z) &= -z^{-2} M'_N(\ln z) + z^{-2} M''_N(\ln z), \\ P''_N(1) &= -M'_N(0) + M''_N(0) = -E[N] + E[N^2] = E[N(N-1)]. \end{aligned}$$