MA (ST) 413 Assignment 3
Solutions

1. (Problem 5.1) Since $Y = \theta X$, we have

$$F_Y(y) = F_X(y/\theta) = 1 - (1 + y/\theta)^{-\alpha} = 1 - \left(\frac{\theta}{\theta + y}\right)^{\alpha}.$$

This is the cdf of the Pareto distribution. The pdf is given by

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{\alpha\theta^\alpha}{(\theta + y)^{\alpha+1}}.$$

2. (Problem 5.3) Inverse:

$$F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-1}}\right)^{\alpha}\right] = \left(\frac{\theta y}{\theta y + 1}\right)^{\alpha}.$$

This is the inverse Pareto distribution with $\tau = \alpha$ and $\theta = 1/\theta$.

Transformed:

$$F_Y(y) = 1 - \left(\frac{\theta}{\theta + y^{\tau}}\right)^{\alpha}.$$

This is the Burr distribution with $\alpha = \alpha, \gamma = \tau$ and $\theta = \theta^{\frac{1}{\tau}}$.

Inverse transformed:

$$F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{\theta + y^{\tau}}\right)^{\alpha}\right] = \left[\frac{(y\theta^{-\frac{1}{\tau}})^{-\tau}}{1 + (y\theta^{-\frac{1}{\tau}})^{-\tau}}\right]^{\alpha}.$$

This is the inverse Burr distribution with $\tau = \alpha, \gamma = -\tau$ and $\theta = \theta^{\frac{1}{\tau}}$.

3. (Problem 5.5) Let Φ be the cdf of the standard normal distribution with mean 0 and variance 1. Then the cdf of a lognormal distribution with parameters μ and σ can be written as

$$F_Y(y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right).$$

Therefore, the cdf of Z is

$$F_Z(z) = F_Y(z/\theta) = \Phi\left(\frac{\ln(z/\theta) - \mu}{\sigma}\right) = \Phi\left(\frac{\ln z - (\mu + \ln\theta)}{\sigma}\right),$$

which is still a lognormal distribution with new parameters $\ln\theta + \mu$ and σ.

4. (Problem 5.9) Since the hazard rate function for the conditional distribution of X is θ, by virtue of the formula $S(x) = e^{-\int_0^x h(t)dt}$, we can get

$$S_{(X|\Theta)}(x|\theta) = e^{-\int_0^x \theta dt} = e^{-\theta x}, \quad x > 0.$$

The pdf of Θ is given by

$$f_\Theta(\theta) = \frac{1}{11-1} = 0.1, \quad x \in [1, 11].$$

Therefore, the unconditional survival function of X is given by

$$S_X(x) = \int S_{(X|\Theta)}(x|\theta) f_\Theta(\theta) d\theta = \int_1^{11} e^{-\theta x} \cdot 0.1 d\theta = \frac{1}{10x}(e^{-x} - e^{-11x}).$$

Therefore, we have

$$S_X(0.5) = \frac{1}{5}(e^{-.5} - e^{-5.5}) = 0.1205.$$

5. (Problem 5.13) It follows from Example 5.7 (see page 68) with $\alpha = 1$ that $S_X(x) = (1 + \theta x^\gamma)^{-1}$. This is a loglogistic distribution with the usual parameter θ (as in Appendix A) replaced by $\theta^{-\frac{1}{\gamma}}$.

6. (Problem 5.17) Let $f_1(x)$ and $f_2(x)$ be the pdf's of the uniform distribution on $[0, 1000]$ and the exponential distribution on $(1000, \infty)$ with mean $1/\theta$. Then we have

$$f_1(x) = \frac{1}{1000}, \quad 0 \le x \le 1000; \quad f_2(x) = \frac{1}{\theta} e^{-\frac{x-1000}{\theta}}, \quad x > 1000.$$

Using the definition of a spliced model, we have

$$f_X(x) = \begin{cases} c_1 f_1(x), & 0 \le x \le 1000, \\ (1-c_1) f_2(x), & x > 1000. \end{cases}$$

We require that the above function is continuous at $x = 1000$, so we have

$$\frac{c_1}{1000} = \frac{1-c_1}{\theta} e^{-\frac{x-1000}{\theta}} |_{x=1000},$$

Solve it, and we can get

$$c_1 = \frac{1000}{1000+\theta}.$$

Therefore, the pdf $f_X(x)$ is

$$f_X(x) = \begin{cases} \frac{1}{1000+\theta}, & 0 \le x \le 1000, \\ \frac{1}{1000+\theta} e^{-\frac{x-1000}{\theta}}, & x > 1000. \end{cases}$$

7. (Problem 5.19) Let X and Y are loss r.v.'s for 1993 and 1994, respectively. Then we have $Y = 1.1X$. Therefore, we have

$$Pr(Y > 2.2) = 1 - F_Y(2.2) = 1 - F_X(2.2/1.1) = 1 - F_X(2).$$

By virtue of $f_X(x) = 3x^{-4}, x \ge 1$, we can get

$$F_X(x) = \int_1^x 3t^{-4} dt = 1 - x^{-3}.$$

So we have

$$Pr(Y > 2.2) = 1 - F_X(2) = 1 - (1 - 2^{-3}) = 0.125.$$