MA (ST) 413 Assignment 2
Solutions

1. (Problem 3.25) (a) While the Weibull distribution has all positive moments, for the inverse Weibull moments exist only for $k < \tau$. Therefore, by this criteria, the inverse Weibull distribution has a heavier tail.

   (b) For the inverse Weibull, the pdf is given

   $$f_1(x) = \tau_1 \theta_1^{\tau_1} x^{-\tau_1 - 1} e^{-\left(\frac{\theta_1}{x}\right)^{\tau_1}}.$$

   For the Weibull distribution, the pdf is given by

   $$f_2(x) = \tau_2 \theta_2^{-\tau_2} x^{\tau_2 - 1} e^{-\left(\frac{x}{\theta_2}\right)^{\tau_2}}.$$

   Therefore, we have

   $$\frac{f_1(x)}{f_2(x)} \propto \frac{\tau_1 \theta_1^{\tau_1} x^{-\tau_1-1} e^{-\left(\frac{\theta_1}{x}\right)^{\tau_1}}}{\tau_2 \theta_2^{-\tau_2} x^{\tau_2-1} e^{-\left(\frac{x}{\theta_2}\right)^{\tau_2}}} \propto x^{-\tau_1-\tau_2} e^{-\left(\frac{\theta_1}{x}\right)^{\tau_1} + \left(\frac{x}{\theta_2}\right)^{\tau_2}} \to \infty.$$

   Thus the inverse Weibull has a heavier tail than the Weibull.

   (c)The hazard rate function for the inverse Weibull is

   $$h_1(x) = \frac{\tau \theta^\tau x^{-\tau-1} e^{-\left(\frac{\theta}{x}\right)^\tau}}{1 - e^{\left(\frac{\theta}{x}\right)^\tau}} \propto \frac{1}{x^{\tau+1}[e^{(\theta/x)^\tau} - 1]}$$

   The derivative of the denominator is

   $$(\tau+1)x^\tau[e^{(\theta/x)^\tau} - 1] + x^{\tau+1} e^{(\theta/x)^\tau} \theta^\tau (-\tau) x^{-\tau-1}.$$

   and the limiting value of this expression is $\theta^\tau > 0$. Therefore, in the limit, the denominator is increasing and thus the hazard rate is decreasing.

   The hazard rate of the Weibull is

   $$h(x) = \frac{\tau \theta^\tau x^{-\tau-1} e^{-\left(\frac{x}{\theta}\right)^\tau}}{e^{-\left(\frac{x}{\theta}\right)^\tau}} = \tau \theta^\tau x^{-\tau-1},$$

   which is clearly increasing when $\tau > 1$, constant when $\tau = 1$ and decreasing when $\tau < 1$. Therefore, the inverse Weibull distribution has a heavier tail.

2. (Problem 3.28) By definition, we have

   $$f_Y(y) = \frac{S_X(y)}{E[X]}.$$

   Therefore, we can get

   $$\begin{aligned} M_Y(y) &= E[e^{tY}] \\ &= \int_0^\infty e^{ty} f_Y(y) dy \\ &= \frac{1}{E[X]} \int_0^\infty e^{ty} S_X(y) dy. \end{aligned}$$

Using integration by parts, we can get

$$\begin{aligned}
\int_0^\infty e^{ty} S_X(y)dy &= \frac{1}{t}e^{ty}S_X(y)\Big|_{y=0}^{y=\infty} - \frac{1}{t}\int_0^\infty e^{ty}S'_X(y)dy \\
&= -\frac{1}{t} + \frac{1}{t}\int_0^\infty e^{ty}f_X(y)dy \\
&= \frac{1}{t}(M_X(t)-1).
\end{aligned}$$

Therefore, we have

$$M_Y(y) = \frac{M_X(t)-1}{tE[X]}.$$

3. (Problem 3.29) (a) Using integration by parts, we have

$$\begin{aligned}
S(x) &= \int_x^\infty (1+2t^2)e^{-2t}dt \\
&= -(1+t+t^2)e^{-2t}|_{t=x}^{t=\infty} \\
&= (1+x+x^2)e^{-2x}, \quad x \geq 0.
\end{aligned}$$

(b) From the above result, we can get

$$f(x) = -\frac{d}{dx}S(x) = (1+2x^2)e^{-2x}, \quad \forall x \geq 0.$$

Therefore, we have

$$h(x) = \frac{f(x)}{S(x)} = \frac{1+2x^2}{1+x+x^2}.$$

(c) For any $y \geq 0$, using integration by parts, we have

$$\begin{aligned}
\int_y^\infty S(t)dt &= \int_y^\infty (1+t+t^2)e^{-2t}dt \\
&= -1(1+t+\frac{1}{2}t^2)e^{-2t}|_{t=y}^{t=\infty} = (1+y+\frac{1}{2}y^2)e^{-2y}.
\end{aligned}$$

Thus, we can get

$$S_e(x) = \frac{\int_x^\infty S(t)dt}{\int_0^\infty S(t)dt} = (1+x+\frac{1}{2}x^2)e^{-2x}, \quad x \geq 0.$$

(d) Using the results of (a) and (c), we can get

$$e(x) = \frac{\int_x^\infty S(t)dt}{S(x)} = \frac{1+x+\frac{1}{2}x^2}{1+x+x^2}.$$

(e) Using the results of (b) and (d) and applying the L'Hopital's rule, we can get

$$\lim_{x\to\infty} h(x) = 2, \quad \lim_{x\to\infty} e(x) = \frac{1}{2}$$

(f) Using the chain rule and the result of (d), we have

$$e'(x) = -\frac{x + \frac{1}{2}x^2}{(1+x+x^2)^2} \leq 0.$$

Therefore, $e(x)$ is strictly decreasing for $x > 0$. On the other hand, using the result of (b), we can get

$$h(0) = 1, \quad h(\frac{1}{2}) = \frac{6}{7}, \quad h(\infty) = 2.$$

Therefore, $h(x)$ is not a strictly decreasing or increasing function.

4. (Problem 4.7) Let Y be the r.v. standing for a randomly selected claim. According to the assumptions, we have

$$F_Y(y) = 0.75F_{X_1}(y) + 0.25F_{X_2}(y) = 0.75\Phi\left(\frac{y-3000}{1000}\right) + 0.25\Phi\left(\frac{y-4000}{1000}\right),$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution $N(0,1)$. Therefore, from the normal distribution table, we can get:

$$\begin{aligned} Pr(Y > 5000) &= 1 - Pr(Y \leq 5000) = 1 - F_Y(5000) \\ &= 1 - \left[0.75\Phi\left(\frac{5000-3000}{1000}\right) + 0.25\Phi\left(\frac{5000-4000}{1000}\right)\right] \\ &= 1 - [0.75\Phi(2) + 0.25\Phi(1)] \\ &= 1 - [0.75 \cdot 0.9772 + 0.25 \cdot 0.8413] \\ &= 0.0568. \end{aligned}$$

5. (Problem 4.11) Let $Y = cX$ and assume that X follows an inverse Gaussian distribution with parameters μ, θ. Then, we can get

$$\begin{aligned} F_Y(y) &= F_X(y/c) \\ &= \Phi\left[\frac{y/c - \mu}{\mu}\left(\frac{\theta}{y/c}\right)^{\frac{1}{2}}\right] + \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-\frac{y/c+\mu}{\mu}\left(\frac{\theta}{y/c}\right)^{\frac{1}{2}}\right] \\ &= \Phi\left[\frac{y - c\mu}{c\mu}\left(\frac{c\theta}{y}\right)^{\frac{1}{2}}\right] + \exp\left(\frac{2c\theta}{c\mu}\right)\Phi\left[-\frac{y+c\mu}{c\mu}\left(\frac{c\theta}{y}\right)^{\frac{1}{2}}\right] \end{aligned}$$

Therefore, Y follows an inverse Gaussian with parameters $c\mu$ and $c\theta$. Therefore, inverse Gaussian is a scale family. In addition, because both θ and μ change, there is no scale parameters.

6. The pdf of a gamma distributed r.v. X with parameters α, θ is $f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}$. Therefore,

$$Pr(X \leq 12) = \int_0^{12} \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}dx.$$

Let $t = \frac{x}{\theta}$. By subtitution, we have

$$\begin{aligned} Pr(X \leq 12) &= \int_0^{12} \frac{x^{\alpha-1} e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)} dx = \int_0^{\frac{12}{\theta}} \frac{(\theta t)^{\alpha-1} e^{-t}}{\theta^\alpha \Gamma(\alpha)} \cdot \theta dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\frac{12}{\theta}} t^{\alpha-1} e^{-t} dt \\ &= \Gamma(\alpha; \frac{12}{\theta}). \end{aligned}$$

Since $\alpha = 3, \theta = 2$, we can get $Pr(X \leq 12) = \Gamma(3; 6)$.