MA(ST) 413 Assignment 1
Solutions

1. (a.)

$$F(x) = \int_{-\infty}^{x} f(x) = \begin{cases} 1, & \text{if } x > 2, \\ \frac{1}{2}x, & \text{if } 0 \le x \le 2, \\ 0, & \text{if } x < 0. \end{cases}$$

$$S(x) = 1 - F(x) = \begin{cases} 0, & \text{if } x > 2, \\ 1 - \frac{1}{2}x, & \text{if } 0 \le x \le 2, \\ 1, & \text{if } x < 0. \end{cases}$$

$$h(x) = \frac{f(x)}{S(x)} = \begin{cases} \frac{1}{2-x}, & \text{if } 0 \le x \le 2, \\ 0, & \text{if } x < 0. \end{cases}$$

(b.)

$$EX = \int_0^2 \frac{1}{2} x dx = 1.$$

$$E[X^2] = \int_0^2 \frac{1}{2} x^2 dx = \frac{4}{3}.$$

$$Var[X] = E[X^2] - [EX]^2 = \frac{1}{3}.$$

(c.) By definition, we should have

$$0.8 = Pr(X \le \pi_{0.8}) = F(\pi_{0.8}) = \frac{1}{2}\pi_{0.8}.$$

Thus, we can get $\pi_{0.8} = 1.6$.

2. (a.)

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$$

$$S(x) = 1 - F(x) = \begin{cases} e^{-\lambda x}, & \text{if } x \ge 0, \\ 1, & \text{if } x < 0. \end{cases}$$

$$h(x) = \frac{f(x)}{S(x)} = \begin{cases} \lambda, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$$

(b.) By definition and using integration by parts, we can get

$$EX = \int_0^\infty x \lambda e^{-\lambda x} dx = \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda},$$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \int_0^\infty 2x e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

$$Var[X] = E[X^2] - [EX]^2 = \frac{1}{\lambda^2}.$$

(c.) By definition, we have

$$0.4 = F(\pi_{0.4}) = 1 - e^{-\lambda \pi_{0.4}}.$$

Solve it, and we can get $\pi_{0.4} = -\frac{1}{\lambda} \ln 0.6$.

3. Let X be a random variable standing for the benefit. Then the probability function for X is

$$Pr(X = 3000) = 0.4, \quad Pr(X = 2000) = 0.6 \cdot 0.4 = 0.24,$$
$$Pr(X = 1000) = 0.6^2 \cdot 0.4 = 0.144,$$
$$Pr(X = 0) = 1 - Pr(X = 3000) - Pr(X = 2000) - Pr(X = 1000) = 0.216.$$

Thus, by definition, we have

$$EX = 0.4 \cdot 3000 + 0.24 \cdot 2000 + 0.144 \cdot 1000 + 0.216 \cdot 0 = 1824.$$

So the expected benefit under the policy is \$1824.

4. From the pgf of X, we can get

$$p_0 = Pr(X = 0) = P(0) = 0.6^4 = 0.1296,$$
$$p_1 = Pr(X = 1) = P'(0) = 4 \cdot 0.6^3 \cdot 0.4 = 0.3456,$$
$$p_2 = Pr(X = 2) = \frac{P''(0)}{2!} = 2 \cdot 3 \cdot 0.6^2 \cdot 0.4^2 = 0.3456,$$
$$p_3 = Pr(X = 3) = \frac{P^{(3)}(0)}{3!} = 4 \cdot 0.6 \cdot 0.4^3 = 0.1536,$$
$$p_4 = Pr(X = 4) = \frac{P^{(4)}(0)}{4!} = 0.4^4 = 0.0256,$$
$$Pr(X \geq 5) = 0.$$

In addition, we can get

$$EX = P'(1) = 4 \cdot 0.4 = 1.6, \quad E[X^2] = P'(1) + P''(1) = 1.6 + 1.92 = 3.52,$$
$$Var[X] = E[X^2] - [EX]^2 = 0.96.$$

5. (Problem 3.19) The cdf for the Gamma distribution is

$$F_X(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha.$$

Therefore, the two percentiles imply that

$$0.1 = F_X(\theta - k) = 1 - \left(\frac{\theta}{\theta + \theta - k}\right)^\alpha,$$
$$0.9 = F_X(5\theta - 3k) = 1 - \left(\frac{\theta}{\theta + 5\theta - 3k}\right)^\alpha.$$

Therefore, we can get

$$0.9 = \left(\frac{\theta}{2\theta - k}\right)^\alpha,$$
$$0.1 = \left(\frac{\theta}{3(2\theta - k)}\right)^\alpha.$$

Dividing the first equation by the second equation, we can get

$$\frac{0.9}{0.1} = \left(\frac{3(2\theta - k)}{2\theta - k}\right)^{\alpha} = 3^{\alpha}.$$

Therefore, we can get

$$\alpha = \frac{\ln 9}{\ln 3} = 2.$$

6. (Problem 3.24) From the sample mean and standard variation, we can get

$$E[X] = 1300, \quad Var(X) = 400^2 = 160000.$$

Therefore, we can get

$$\begin{aligned}
&E[S] = 2500 \cdot E[X] = 2500 \times 1300 = 3,250,000, \\
&Var(S) = 2500 \cdot Var(X) = 2500 \times 160000 = 400,000,000, \\
&\sqrt{Var(S)} = \sqrt{400000000} = 20,000.
\end{aligned}$$

By virtue of the Central Limit Theorem, we know that

$$\frac{S - E[S]}{\sqrt{Var(X)}} \sim N(0,1).$$

Therefore, we can get

$$\begin{aligned}
Pr(S > 1.01E[S]) &= Pr\left(\frac{S - E[S]}{\sqrt{Var(X)}} > \frac{1.01E[S] - E[S]}{\sqrt{Var(X)}}\right) \\
&= Pr\left(Z > \frac{0.01 \cdot 3250000}{20000}\right) \\
&= Pr(Z > 1.625) \\
&= 0.052,
\end{aligned}$$

where Z is a standard normal r.v.