Γ-Function and B-Function (Beta-Function)

1. Definition. The Γ-function is defined by

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t} dt, \quad \forall x > 0.$$

The B-function is defined by

$$B(x, y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \forall x, y > 0.$$

2. Properties and Relations.

- $\Gamma(1) = 1$.
- For any $x > 0$, we have $\Gamma(x+1) = x\Gamma(x)$.
- If $n > 0$ is an integer, then $\Gamma(n+1) = n!$.
- $B(x, y) = B(y, x), \quad \forall x, y > 0$.
- $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
- $B(x+1, y) = \frac{x}{x+y} B(x, y)$.

3. Incomplete Γ-function and Incomplete Beta-function.
In probability theory, the incomplete Γ-function $\Gamma(\alpha; x)$ and the incomplete Beta-function $B(a, b; x)$ are often used. They are defined by

$$\begin{aligned} \Gamma(\alpha; x) &\equiv \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt, \quad 0 \le x < \infty; \\ B(a, b; x) &\equiv \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad 0 \le x \le 1. \end{aligned}$$

4. Properties.
From the definitions, we can get that both $\Gamma(\alpha; x)$ and $B(a, b; x)$ are increasing functions of x and they satisfy

$$\Gamma(\alpha; 0) = 0, \quad \lim_{x \to \infty} \Gamma(\alpha; x) = 1;$$
$$B(a, b; 0) = 0, \quad B(a, b; 1) = 1.$$