MA 231 Midterm Exam 2 07/26/2010
Solutions

1. (20 points)

   (a) (10 points) Using given information, we have

$$\begin{aligned}
\iint\limits_R 2xe^{xy}dxdy &= \int_0^2 \left(\int_0^3 2xe^{xy}dy\right)dx \\
&= \int_0^2 \left(2e^{xy}\Big|_{y=0}^{y=3}\right)dx \\
&= \int_0^2 \left(2e^{3x}-2\right)dx \\
&= \left(\frac{2}{3}e^{3x}-2x\right)\Big|_{x=0}^{x=2} \\
&= \frac{2}{3}(e^6-7) \\
&= 264.29.
\end{aligned}$$

   (b) (10 points)

$$\begin{aligned}
\int_1^3 \left(\int_x^{3x} x(1+y)dy\right)dx &= \int_1^3 \left(x(y+\frac{1}{2}y^2)\Big|_{y=x}^{y=3x}\right)dx \\
&= \int_1^3 \left(2x^2+4x^3\right)dx \\
&= \left(\frac{2}{3}x^3+x^4\right)\Big|_{x=1}^{x=3} \\
&= 97\frac{1}{3}.
\end{aligned}$$

2. (20 points) Let a be the first term, and r be the common ratio.

   (a) (10 points) It is easy to see that $a=\frac{2}{3}, r=-\frac{1}{3}$. Since $|r|<1$, the series converges. The sum is

$$\frac{a}{1-r}=\frac{\frac{2}{3}}{1-(-\frac{1}{3})}=\frac{1}{2}.$$

   (b) (10 points) We can see that $a=\frac{3}{2^3}, r=\frac{1}{2}$. Since $|r|<1$, the series converges. The sum is

$$\frac{a}{1-r}=\frac{\frac{3}{2^3}}{1-\frac{1}{2}}=\frac{3}{4}.$$

3. (20 points) Let $f(x) = \ln x$. Then we have

$$\begin{aligned} f(x) &= \ln x, \quad f(1) = 0; \\ f'(x) &= \frac{1}{x}, \quad f'(1) = 1; \\ f''(x) &= \frac{-1}{x^2}, \quad f''(1) = -1; \\ f^{(3)}(x) &= \frac{2}{x^3}, \quad f^{(3)}(1) = 2. \end{aligned}$$

Therefore, the 3rd Taylor polynomial at $x = 1$ for $\ln x$ is

$$p_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

To calculate $\ln 1.05$, we calculate $p_3(1.05)$:

$$p_3(1.05) = (0.05) - \frac{(0.05)^2}{2} + \frac{(0.05)^3}{3} = 0.0486.$$

4. (20 points) Start from the Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Replace x by $(-x)$, and we can get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

Replace x by $(2x^2)$, and we can get

$$\begin{aligned} \frac{1}{1+2x^2} &= 1 - 2x^2 + (2x^2)^2 - (2x^2)^3 + \cdots \\ &= 1 - 2x^2 + 4x^4 - 8x^6 + \cdots . \end{aligned}$$

Multiple by x, and we have

$$\frac{x}{1+2x^2} = x - 2x^3 + 4x^5 - 8x^7 + \cdots .$$

5. (10 points) Using the Taylor series of $f(x)$, we can get

$$f^{(3)}(0) = 32 \cdot 3! = 192.$$

6. (10 points) Let $P(t)$ be population at time t. Then we have

$$P'(t) = 0.05P(t).$$

This is an exponential growth model. The solution is

$$P(t) = Ce^{0.05t}.$$

Using the initial value condition $P(0) = 1$, we can get that $C = 1$. So the solution is

$$P(t) = e^{0.05t}.$$

We want to find a time T such that

$$P(T) = 2.$$

That is

$$e^{0.05T} = 2.$$

Solve it to get

$$T = \frac{\ln 2}{0.05} = 13.86.$$

size is 1 million, how long does it take for the population size to become 2 million?

-The End-