Optimal Stopping Problem for Stochastic Differential Equations with Random Coefficients*

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Abstract

An optimal stopping problem for stochastic differential equations with random coefficients is considered. Dynamic programming principle leads to a Hamiltion-Jacobi-Bellman equation which, for the current case, is a backward stochastic partial differential variational inequality (BSPDVI, for short) for the value function. Well-posedness of such a BSPDVI is established and a verification theorem is proved.

Keywords. optimal stopping, random coefficients, dynamic programming principle, backward stochastic partial differential variational inequality, verification theorem.

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1 Introduction.

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $\mathcal{S}[0,T]$ be the set of all $\mathbb{F}$-stopping times taking values in $[0,T]$. For any $\tau_1, \tau_2 \in \mathcal{S}[0,T]$ with $\tau_1 \leq \tau_2$ almost surely, and $\mathbb{P}\{\tau_1 < \tau_2\} > 0$, let

$$\left\{\begin{array}{l}\mathcal{S}[\tau_1,\tau_2] \stackrel{\Delta}{=} \Big\{\tau \in \mathcal{S}[0,T] \bigm| \tau_1 \leq \tau \leq \tau_2, \text{ a.s. } \Big\},\\ \mathcal{S}(\tau_1,\tau_2] \stackrel{\Delta}{=} \Big\{\tau \in \mathcal{S}[\tau_1,\tau_2] \bigm| \tau_1 < \tau, \text{ a.s. on } \{\tau_1<\tau_2\}\Big\},\\ \mathcal{S}[\tau_1,\tau_2) \stackrel{\Delta}{=} \Big\{\tau \in \mathcal{S}[\tau_1,\tau_2] \bigm| \tau < \tau_2, \text{ a.s. on}\{\tau_1<\tau_2\}\Big\},\\ \mathcal{S}(\tau_1,\tau_2) \stackrel{\Delta}{=} \Big\{\tau \in \mathcal{S}[\tau_1,\tau_2] \bigm| \tau_1 < \tau < \tau_2, \text{ a.s. on } \{\tau_1<\tau_2\}\Big\}.\end{array}\right. \tag{1.1}$$

Next, for any $s \in \mathcal{S}[0,T]$ and $p \geq 1$, denote

$$\mathcal{X}_s^p \equiv L^p_{\mathcal{F}_s}(\Omega;\mathbb{R}^n) \stackrel{\Delta}{=} \Big\{\xi : \Omega \to \mathbb{R}^n \bigm| \xi \text{ is } \mathcal{F}_s\text{-measurable}, \mathbb{E}|\xi|^p < \infty\Big\}. \tag{1.2}$$

For any $s \in \mathcal{S}[0,T)$ and $\xi \in \mathcal{X}_s^p$, consider the following stochastic differential equation (SDE, for short):

$$\left\{\begin{array}{ll} dX(t) = b(t,X(t))dt + \sigma(t,X(t))dW(t), & t \in [s,T],\\ X(s) = \xi, \end{array}\right. \tag{1.3}$$

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where $b : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n\times d}$ are given maps. We refer to the above as the *state equation.* Under proper conditions (which will be assumed shortly), the above SDE admits a unique strong solution $X(\cdot) \equiv X(\cdot\,; s, \xi)$. Introduce the following *cost functional*:

$$J_{s,\xi}(\tau) = \mathbb{E}\Big[\int_s^\tau g(t, X(t; s, \xi))dt + h(\tau, X(\tau; s, \xi))\big|\mathcal{F}_s\Big], \qquad \tau \in \mathcal{S}[s,T], \tag{1.4}$$

where $g, h : [0,T] \times \mathbb{R}^n \times \Omega \to [0,\infty)$ are some given non-negative maps satisfying proper conditions. The two terms on the right hand side of (1.4) represent the *running cost* and the *terminal cost*, respectively. We point out that all the involved maps b, σ, g, and h in our discussion are allowed to be random. With the above setting, we can now pose the following optimal stopping problem.

Problem (S). *For given $s \in \mathcal{S}[0,T)$ and $\xi \in \mathcal{X}_s^p$, find the smallest $\bar\tau \in \mathcal{S}[s,T]$ such that*

$$J_{s,\xi}(\bar\tau) = \inf_{\tau\in\mathcal{S}[s,T]} J_{s,\xi}(\tau) \equiv V(s,\xi). \tag{1.5}$$

Any $\bar\tau \in \mathcal{S}[s,T]$ satisfying (1.5) is referred to as an *optimal stopping time*, and the smallest one is referred to as the *smallest optimal stopping time.* We compatibly define

$$V(T,\xi) = h(T,\xi), \qquad \forall \xi \in \mathcal{X}_s^p. \tag{1.6}$$

Random field $V(\cdot\,,\cdot)$ defined by (1.5)–(1.6) is called the *value function* of Problem (S). We point out that for the maps g and h, non-negativity condition can be relaxed to the boundedness from below. On the other hand, it is not hard to see that if $h = 0$ and $g > 0$, then any optimal stopping time of Problem (S) must be the smallest one. But, in general, optimal stopping time of Problem (S) is not necessarily unique (one can modify Example D.14 of [15]). Hence, to be definite, our Problem (S) is to find the smallest optimal stopping time. We also note that, due to that the coefficients are allowed to be random, and our cost functional is defined by a conditional expectation, our value function $V(\cdot\,,\cdot)$ is actually a random field.

In the case that all the coefficients are deterministic, one can prove the dynamic programming principle which leads to a partial differential variational inequality (PDVI, for short), as the corresponding HJB equation for the value function (which is deterministic). Moreover, it can be shown that the value function is the unique viscosity solution to the PDVI. In the case that the diffusion is uniformly non-degenerate, the value function is the (unique) classical solution of the PDVI, provided that some mild smoothness conditions are assumed for the coefficients. On the other hand, one can independently establish the well-posedness of the corresponding PDVI, as well as a verification theorem. These will then provide a solution to the original optimal stopping time problem (See [2], and the references cited therein).

We also note that, by some pure probabilistic approach, one can study optimal stopping time problem for general continuous-time stochastic processes. Optimal stopping time is characterized by means of the so-called Snell's envelope, super martingale, and so on, without using dynamic programming principle. In such an approach, no HJB equation is involved which is natural because no dynamic equation is assumed for the considered stochastic processes ([15]). We refer to [10], [11], [27], [3], [26], [8], [30], [24], [31], [20], [1], [5], [25], [4], [29], [7], [12], for relevant results on stochastic optimal stopping and optimal control problems.

For the problem under our consideration, since we have more structures on the stochastic process (satisfying an SDE, etc.), it is expected to have more detailed characterization on the optimal stopping time. On the other hand, due to the randomness of the coefficients, the usual technique of dynamic programming principle together with theories of PDVIs do not directly apply. In this paper, inspired by [23], we will formally derive the corresponding HJB equation for the value function $V(\cdot\,,\cdot)$, which is now a *backward stochastic partial differential variational inequality* (BSPDVI, for short). Using a result of semilinear backward stochastic partial differential equations (BSPDEs, for short) from [28], together with a standard penalty technique for (deterministic) PDVIs ([13]), we will obtain the well-posedness of our BSPDVI in a certain sense. At the same time, a verification theorem will be established, which says that, under proper conditions, the solution to the BSPDVI coincides with the value function of Problem (S). Then an optimal stopping time can be characterized. See [22] for some results concerning backward stochastic variational inequalities in an abstract framework.

The rest of the paper is organized as follows: Some preliminary results, including certain basic properties of the value function will be presented in Section 2. In Section 3, we will formally derive the BSPDVI, and formally prove a verification theorem. Notions of adapted solutions will be introduced in Section 4. The well-posedness of the BSPDVI will be established in Section 5. Finally, in Section 6, the adapted weak solution of the BSPDVI will be identified as the value function of Problem (S).

2 Some Preliminary Results.

In this section, we are going to present some preliminary results related to the value function $V(\cdot\,,\cdot)$ of Problem (S). To begin with, for any $p \geq 1$, $s \in \mathcal{S}[0,T)$ and $\tau \in \mathcal{S}(s,T]$, we let $L^p_{\mathbb{F}}(\Omega; C([s,\tau];\mathbb{R}^n))$ be the set of all processes $\varphi : [s,\tau] \to \mathbb{R}^n$ having continuous paths and

$$\mathbb{E}\Big[\sup_{t\in[s,\tau]} |\varphi(t)|^p\Big] < \infty.$$

It is clear that $L^p_{\mathbb{F}}(\Omega; C([s,\tau];\mathbb{R}^n))$ is a Banach space. Next, for $p \geq 1$, we denote (recall (1.2))

$$\mathcal{D}^p = \Big\{(s,\xi) \in \mathcal{S}[0,T] \times \mathcal{X}^p_T \;\big|\; s \in \mathcal{S}[0,T],\ \xi \in \mathcal{X}^p_s\Big\}.$$

Now, we introduce the following standing assumption concerning the coefficients of the state equation (1.3).

(H1) *Maps $b : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n\times d}$ are measurable and they satisfy the following:*

(a) *For each $x \in \mathbb{R}^n$, $t \mapsto (b(t,x), \sigma(t,x))$ is $\mathbb{F}$-progressively measurable and for some $p > 1$,*

$$\mathbb{E}\Big(\int_0^T |b(t,0)|dt\Big)^p + \mathbb{E}\Big(\int_0^T |\sigma(t,0)|^2 dt\Big)^{\frac{p}{2}} < \infty. \tag{2.1}$$

(b) *There exists an $L > 0$ such that*

$$\begin{aligned}&|b(t,x,\omega) - b(t,y,\omega)| + |\sigma(t,x,\omega) - \sigma(t,y,\omega)| \leq L|x-y|,\\ &\qquad\qquad \text{a.e. } t \in [0,T],\ \forall x,y \in \mathbb{R}^n,\ \text{a.s. } \omega \in \Omega.\end{aligned} \tag{2.2}$$

Concerning the maps appearing in the cost functional, we introduce the following assumption:

(H2) *Maps $g, h : [0,T] \times \mathbb{R}^n \times \Omega \to [0,\infty)$ are measurable and they satisfy the following:*

(a) *For each $x \in \mathbb{R}^n$, $t \mapsto (g(t,x), h(t,x))$ is $\mathbb{F}$-progressively measurable; for each $x \in \mathbb{R}^n$ and almost surely $\omega \in \Omega$, $t \mapsto h(t,x)$ is continuous, and*

$$\mathbb{E}\Big[\int_0^T g(t,0)dt + \sup_{t\in[0,T]} h(t,0)\Big] < \infty. \tag{2.3}$$

(b) *There exists an $L > 0$ such that*

$$\begin{aligned}&|g(t,x,\omega) - g(t,y,\omega)| + |h(t,x,\omega) - h(t,y,\omega)| \leq L|x-y|,\\ &\qquad\qquad \text{a.e. } t \in [0,T],\ \forall x,y \in \mathbb{R}^n,\ \text{a.s. } \omega \in \Omega,\end{aligned} \tag{2.4}$$

and there exists a continuous nondecreasing function $\rho : [0,\infty) \to [0,\infty)$ with $\rho(0) = 0$ such that

$$|h(t,x,\omega) - h(s,x,\omega)| \le (1+|x|)\rho(|t-s|), \qquad \forall t,s \in [0,T],\ x \in \mathbb{R}^n, \quad \text{a.s.} \tag{2.5}$$

The following result is pretty standard (see [14]).

Proposition 2.1. *Let* (H1) *hold. Then for each $(s,\xi) \in \mathcal{D}^p$, state equation* (1.3) *admits a unique (strong) solution $X(\cdot) \equiv X(\cdot\,; s,\xi) \in L^p_{\mathbb{F}}(\Omega; C([s,T];\mathbb{R}^n))$. Moreover,*

$$\mathbb{E}\Big[\sup_{t\in[s,T]} |X(t;s,\xi)|^p \bigm| \mathcal{F}_s\Big] \le C\Big(1+|\xi|^p\Big), \qquad \forall (s,\xi) \in \mathcal{D}^p, \tag{2.6}$$

$$\mathbb{E}\Big[\sup_{t\in[s,T]} |X(t;s,\xi) - X(t;s,\bar\xi)|^p \bigm| \mathcal{F}_s\Big] \le C|\xi-\bar\xi|^p, \qquad \forall (s,\xi),(s,\bar\xi) \in \mathcal{D}^p, \tag{2.7}$$

and when $p>1$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[s,\tau]} |X(t;s,\xi) - \xi|^{\bar p} \bigm| \mathcal{F}_s\Big] &\le C(1+|\xi|^{\bar p})\Big\{\mathbb{E}\Big[|\tau-s|^{\frac{p\bar p}{2(p-\bar p)}} \bigm| \mathcal{F}_s\Big]\Big\}^{\frac{p-\bar p}{p}},\\ &\forall (s,\xi) \in \mathcal{D}^p, \quad \tau \in \mathcal{S}[s,T], \quad \bar p \in [1,p),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[s\vee\bar s,T]} |X(t;s,\xi) - X(t;\bar s,\xi)|^{\bar p} \bigm| \mathcal{F}_{s\wedge\bar s}\Big] &\le C(1+|\xi|^{\bar p})\Big\{\mathbb{E}\Big[|s-\bar s|^{\frac{p\bar p}{2(p-\bar p)}} \bigm| \mathcal{F}_{s\wedge\bar s}\Big]\Big\}^{\frac{p-\bar p}{p}},\\ &\forall s,\bar s \in \mathcal{S}[0,T], \quad \xi \in \mathcal{X}^p_{s\wedge\bar s}, \quad \bar p \in [1,p),\end{aligned} \tag{2.9}$$

hereafter, $C>0$ represents a generic constant which can be different from line to line.

A simple consequence of the above is that

$$(t, X(t;s,\xi)) \in \mathcal{D}^p, \qquad t \in \mathcal{S}[s,T], \qquad \forall (s,\xi) \in \mathcal{D}^p. \tag{2.10}$$

We also note that if both $s,\bar s \in [0,T]$ are deterministic, then

$$\mathbb{E}\Big[\sup_{t\in[s\vee\bar s,T]} |X(t;s,\xi) - X(t;\bar s,\xi)|^p \bigm| \mathcal{F}_{s\wedge\bar s}\Big] \le C(1+|\xi|^p)|s-\bar s|^{\frac{p}{2}}, \qquad \forall \xi \in \mathcal{X}^p_{s\wedge\bar s}. \tag{2.11}$$

The following proposition collects some basic results concerning the value function $V(\cdot\,,\cdot)$.

Proposition 2.2. *Let* (H1)–(H2) *hold. Then*

(i) *For any $(s,\xi) \in \mathcal{D}^p$ and $\tau \in \mathcal{S}[s,T]$, $J_{s,\xi}(\tau)$ is a well-defined $\mathcal{F}_s$-measurable random variable. Moreover, there exists a $\bar\tau(s,\xi) \in \mathcal{S}[s,T]$ such that*

$$V(s,\xi) \equiv \inf_{\tau\in\mathcal{S}[s,T]} J_{s,\xi}(\tau) = J_{s,\xi}(\bar\tau(s,\xi)). \tag{2.12}$$

Consequently, for any $(s,\xi) \in \mathcal{D}^p$, $V(s,\xi)$ is $\mathcal{F}_s$-measurable.

(ii) *Value function $V(\cdot\,,\cdot)$ satisfies the following:*

$$|V(s,\xi)| \le C(1+|\xi|), \qquad \forall (s,\xi) \in \mathcal{D}^p, \tag{2.13}$$

$$|V(s,\xi) - V(s,\bar\xi)| \le C|\xi-\bar\xi|, \qquad \forall (s,\xi),(s,\bar\xi) \in \mathcal{D}^p, \tag{2.14}$$

and when $p>1$,

$$\begin{aligned}\Big|\mathbb{E}\Big[V(s,\xi) - V(\bar s,\xi) \bigm| \mathcal{F}_{s\wedge\bar s}\Big]\Big| &\le C(1+|\xi|)\Big\{\Big[\mathbb{E}\Big(|s-\bar s|^{\frac{p}{2(p-1)}} \bigm| \mathcal{F}_{s\wedge\bar s}\Big)\Big]^{\frac{p-1}{p}}\\ &+\mathbb{E}\Big[\rho(|s-\bar s|) + |s-\bar s| \bigm| \mathcal{F}_{s\wedge\bar s}\Big]\Big\}, \quad \forall s,\bar s \in \mathcal{S}[0,T],\ \xi \in \mathcal{X}^p_{s\wedge\bar s}.\end{aligned} \tag{2.15}$$

(iii) *For any* $s\in\mathcal{S}[0,T)$ *and any* $\varphi(\cdot)\in L^1_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$*, the map* $t\mapsto V(t,\varphi(t))$ *is* $\mathbb{F}$*-adapted on* $[s,T]$*. In particular, for any* $(s,\xi)\in\mathcal{D}^p$*, the map* $t\mapsto V(t,X(t;s,\xi))$ *is* $\mathbb{F}$*-adapted.*

Proof. (i) By Proposition 2.1 and (H2), we see that for any fixed $(s,\xi)\in\mathcal{D}^p$,

$$|J_{s,\xi}(\tau)|\le\mathbb{E}\Big[\int_s^\tau|g(r,X(r;s,\xi))|dr+|h(\tau,X(\tau;s,\xi))|\;\Big|\;\mathcal{F}_s\Big]\le C(1+|\xi|),\qquad\tau\in\mathcal{S}[s,T].$$

Hence, $J_{s,\xi}(\tau)$ is a well-defined $\mathcal{F}_s$-measurable random variable. Next, it is clear that $t\mapsto J_{s,\xi}(t)$ is continuous. Thus, by Theorem D.12 of [15] (see also [8]), with a minor modification, we have the existence of an optimal stopping time $\bar\tau(s,\xi)$ for Problem (S).

(ii) For any $(s,\xi),(s,\bar\xi)\in\mathcal{D}^p$, by (H2) and Proposition 2.1, we can get

$$\begin{aligned}|J_{s,\xi}(\theta)-J_{s,\bar\xi}(\theta)|&\le\mathbb{E}\Big[\int_s^\theta|g(r,X(r;s,\xi))-g(r,X(r;s,\bar\xi))|dr\\&\qquad+|h(\theta,X(\theta;s,\xi))-h(\theta,X(\theta;s,\bar\xi))|\;\Big|\;\mathcal{F}_s\Big]\\&\le C\mathbb{E}\Big[\sup_{t\in[s,\theta]}|X(t;s,\xi)-X(t;s,\bar\xi)|\;\Big|\;\mathcal{F}_s\Big]\le C|\xi-\bar\xi|,\quad\forall\theta\in\mathcal{S}[s,T],\end{aligned}\tag{2.16}$$

with $C>0$ being an absolute constant. Hence, (2.14) follows. Next, let $s,\bar s\in\mathcal{S}[0,T]$, $\xi\in\mathcal{X}^p_{s\wedge\bar s}$, and $\theta\in\mathcal{S}[s\wedge\bar s,T]$. Observe the following:

$$\begin{aligned}&\Big|\mathbb{E}\Big[J_{s,\xi}(s\vee\theta)-J_{\bar s,\xi}(\bar s\vee\theta)\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big]\Big|\\&=\Big|\mathbb{E}\Big[h(s\vee\theta,X(s\vee\theta;s,\xi))-h(\bar s\vee\theta,X(\bar s\vee\theta;\bar s,\xi))\\&\qquad+\int_s^{s\vee\theta}g(t,X(t;s,\xi))dt-\int_{\bar s}^{\bar s\vee\theta}g(t,X(t;\bar s,\xi))dt\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big]\Big|\\&=\Big|\mathbb{E}\Big\{I_{(s\vee\bar s\le\theta)}\Big[h(\theta,X(\theta;s,\xi))-h(\theta,X(\theta;\bar s,\xi))\Big]\\&\qquad+I_{(s<\theta<\bar s)}\Big[h(\theta,X(\theta;s,\xi))-h(\bar s,\xi)\Big]+I_{(\bar s<\theta<s)}\Big[h(s,\xi)-h(\theta,X(\theta;\bar s,\xi))\Big]\\&\qquad+I_{(s\vee\bar s\le\theta)}\Big[\int_s^{s\vee\bar s}g(t,X(t;s,\xi))dt-\int_{\bar s}^{s\vee\bar s}g(t,X(t;\bar s,\xi))dt\\&\qquad+\int_{s\vee\bar s}^\theta\Big(g(t,X(t;s,\xi))dt-g(t,X(t;\bar s,\xi))\Big)dt\Big]\\&\qquad+I_{(s<\theta<\bar s)}\int_s^\theta g(t,X(t;s,\xi))dt-I_{(\bar s<\theta<s)}\int_{\bar s}^\theta g(t,X(t;\bar s,\xi))dt\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big\}\Big|\\&\le\mathbb{E}\Big\{I_{(s\vee\bar s\le\theta)}L|X(\theta;s,\xi)-X(\theta;\bar s,\xi)|+I_{(s<\theta<\bar s)}\Big[L|X(\theta;s,\xi)-\xi|+(1+|\xi|)\rho(|s-\bar s|)\Big]\\&\qquad+I_{(\bar s<\theta<s)}\Big[L|X(\theta;\bar s,\xi)-\xi|+(1+|\xi|)\rho(|s-\bar s|)\Big]\\&\qquad+C\Big(1+\sup_{t\in[s,T]}|X(t;s,\xi)|+\sup_{t\in[\bar s,T]}|X(t;\bar s,\xi)|\Big)|s-\bar s|\\&\qquad+I_{(s\vee\bar s\le\theta)}\int_{s\vee\bar s}^\theta L|X(t;s,\xi)-X(t;\bar s,\xi)|dt\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big\}\\&\le C\Big\{(1+|\xi|)\Big[\mathbb{E}\Big(|s-\bar s|^{\frac{p}{2(p-1)}}\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big)\Big]^{\frac{p-1}{p}}+(1+|\xi|)\mathbb{E}\Big[\rho(|s-\bar s|)\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big]\\&\qquad+(1+|\xi|)\mathbb{E}\Big[|s-\bar s|\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big]+(1+|\xi|)\Big[\mathbb{E}\Big(|s-\bar s|^{\frac{p}{2(p-1)}}\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big)\Big]^{\frac{p-1}{p}}\Big\}\\&\le C(1+|\xi|)\Big\{\Big[\mathbb{E}\Big(|s-\bar s|^{\frac{p}{2(p-1)}}\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big)\Big]^{\frac{p-1}{p}}+\mathbb{E}\Big[\rho(|s-\bar s|)+|s-\bar s|\;\Big|\;\mathcal{F}_{s\wedge\bar s}\Big]\Big\}.\end{aligned}\tag{2.17}$$

Hence, taking $\theta=\bar\tau(\bar s,\xi)$, we obtain (note $\bar\tau(\bar s,\xi)\ge \bar s$)

$$\begin{aligned}&\mathbb{E}\Big[V(s,\xi)-V(\bar s,\xi)\bigm|\mathcal{F}_{s\wedge\bar s}\Big]\le\mathbb{E}\Big[J_{s,\xi}(s\vee\bar\tau(\bar s,\xi))-J_{\bar s,\xi}(\bar\tau(\bar s,\xi))\bigm|\mathcal{F}_{s\wedge\bar s}\Big]\\&\le C(1+|\xi|)\Big\{\Big[\mathbb{E}\Big(|s-\bar s|^{\frac{p}{2(p-1)}}\bigm|\mathcal{F}_{s\wedge\bar s}\Big)\Big]^{\frac{p-1}{p}}+\mathbb{E}\Big[\rho(|s-\bar s|)+|s-\bar s|\bigm|\mathcal{F}_{s\wedge\bar s}\Big]\Big\}.\end{aligned}\tag{2.18}$$

Exchanging the roles of s and $\bar s$, we obtain (2.15).

(iii) is clear. □

3 Principle of Optimality and BSPDVI.

We now would like to formally derive the equation that the value function $V(\cdot\,,\cdot)$ should satisfy. To this end, we first state the following principle of optimality.

Theorem 3.1. *Let* (H1)–(H2) *hold.*

(i) *For any* $(s,\xi)\in\mathcal{D}^p$,

$$V(s,\xi)\le h(s,\xi),\qquad \text{a.s. },\tag{3.1}$$

and

$$V(s,\xi)\le\inf_{\tau\in\mathcal{T}[s,T]}\mathbb{E}\Big[\int_s^\tau g(r,X(r;s,\xi))dr+V(\tau,X(\tau;s,\xi))\big|\mathcal{F}_s\Big],\quad\text{a.s.}\tag{3.2}$$

(ii) *For any* $(s,\xi)\in\mathcal{D}^p$, *if* $\bar\theta\in\mathcal{S}[s,T]$ *is an optimal stopping time of Problem (S) for the initial point* (s,ξ), *then*

$$V(\bar\theta,X(\bar\theta;s,\xi))=h(\bar\theta,X(\bar\theta;s,\xi)),\qquad\text{a.s.}\tag{3.3}$$

Hence, the following is the smallest optimal stopping time of Problem (S) corresponding to (s,ξ):

$$\bar\tau(s,\xi)=\inf\Big\{t\in[s,T]\bigm|V(t,X(t;s,\xi))=h(t,X(t;s,\xi))\Big\}.\tag{3.4}$$

Moreover,

$$\mathbb{P}\Big(\{\bar\tau(s,\xi)>s\}\Delta\{V(s,\xi)<h(s,\xi)\}\Big)=0,\tag{3.5}$$

where $A\Delta B=(A\setminus B)\cup(B\setminus A)$, *for any* $A,B\in\mathcal{F}$, *and*

$$\begin{aligned}V(\theta,X(\theta;s,\xi))=\mathbb{E}\Big[\int_\theta^\tau g(r,X(r;s,\xi))dr+V(\tau,X(\tau;s,\xi))\big|\mathcal{F}_\theta\Big],\\ \forall\theta\in\mathcal{S}[s,\bar\tau(s,\xi)],\quad\tau\in\mathcal{S}[\theta,\bar\tau(s,\xi)],\quad\text{a.s.}\end{aligned}\tag{3.6}$$

The above results are basically known (see [8]). For readers's convenience, we sketch a proof in the appendix.

Note that (3.5) tells us the following: Up to a $\mathbb{P}$-null set, one has

$$\{\bar\tau(s,\xi)>s\}=\{V(s,\xi)<h(s,\xi)\}.\tag{3.7}$$

Consequently, up to a $\mathbb{P}$-null set, the following holds:

$$\{\bar\tau(s,\xi)=s\}=\{V(s,\xi)=h(s,\xi)\}.\tag{3.8}$$

On the other hand, (3.2) implies that

$$\begin{aligned}V(\theta,X(\theta;s,\xi))+\int_s^\theta g(r,X(r;s,\xi))dr\le\mathbb{E}\Big[V(\tau,X(\tau;s,\xi))+\int_s^\tau g(r,X(r;s,\xi))dr\big|\mathcal{F}_\theta\Big],\\ \forall\theta\in\mathcal{S}[s,T],\ \tau\in\mathcal{S}[\theta,T].\end{aligned}\tag{3.9}$$

This means that

$$\theta \mapsto V(\theta, X(\theta; s, \xi)) + \int_s^\theta g(r, X(r; s, \xi)) dr$$

is an $\mathbb{F}$-submartingale on $[s, T]$. Likewise, (3.6) implies that

$$\begin{aligned} V(\theta, X(\theta; s, \xi)) + \int_s^\theta g(r, X(r; s, \xi)) dr = \mathbb{E}\Big[V(\tau, X(\tau; s, \xi)) + \int_s^\tau g(r, X(r; s, \xi)) dr \Big| \mathcal{F}_\theta\Big], \\ \forall \theta \in \mathcal{S}[s, \bar{\tau}(s, \xi)],\ \tau \in \mathcal{S}[\theta, \bar{\tau}(s, \xi)], \end{aligned} \tag{3.10}$$

which means that

$$\theta \mapsto V(\theta, X(\theta; s, \xi)) + \int_s^\theta g(r, X(r; s, \xi)) dr$$

is an $\mathbb{F}$-martingale on $[s, \bar{\tau}(s, \xi)]$.

Next, we would like to derive the Hamilton-Jacobi-Bellman equation for the value function $V(\cdot\,,\cdot)$. To this end, let us first make a convention: for any differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$, with $m > 1$, the gradient $f_x : \mathbb{R}^n \to \mathbb{R}^{m\times n}$, and for $m = 1$, $f_x : \mathbb{R}^n \to \mathbb{R}^n$. Now, we recall a special case of the Itô–Kunita's formula ([16], [23]).

Theorem 3.2. *Let $F : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfy the following:*

(1) *$(t, x) \mapsto F(t, x, \omega)$ is continuous almost surely;*

(2) *$x \mapsto F(t, x, \omega)$ is C^2, for each $t \in [0, T]$, almost surely;*

(3) *For each $x \in \mathbb{R}^n$, $t \mapsto F(t, x, \cdot)$ is a continuous semi-martingale with*

$$F(t, x) = F(0, x) + \int_0^t q^0(r, x) dr + \int_0^t \langle\, q(r, x), dW(r)\,\rangle, \qquad (t, x) \in [0, T] \times \mathbb{R}^n,$$

for some $q^0(\cdot)$ and $q(\cdot)$ satisfying the following: For each $x \in \mathbb{R}^n$, $t \mapsto (q^0(t, x), q(t, x))$ is $\mathbb{F}$-adapted, taking values in $\mathbb{R} \times \mathbb{R}^d$, and for almost all $(t, \omega) \in [0, T] \times \Omega$, $x \mapsto q(t, x)$ is C^1. Then

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) + \int_0^t \Big\{ & q^0(r, X(r)) + \langle\, b(r, X(r)), F_x(r, X(r))\,\rangle \\ & + \frac{1}{2}\mathrm{tr}\Big[\sigma(r, X(r))\sigma(r, X(r))^T F_{xx}(r, X(r))\Big] + \mathrm{tr}\Big[\sigma(r, X(r)) q_x(r, X(r))\Big]\Big\} dr \\ & + \int_0^t \langle\, q(r, X(r)) + \sigma(r, X(r))^T F_x(r, X(r)), dW(r)\,\rangle\,. \end{aligned} \tag{3.11}$$

According to our convention, q_x is taking values in $\mathbb{R}^{d\times n}$, and F_x is taking values in $\mathbb{R}^n$. Now, for any $(s, \xi) \in \mathcal{D}^p$, suppose $\bar{\tau}(s, \xi)$ is the corresponding minimum optimal stopping time. Suppose the value function $V(\cdot\,,\cdot)$ admits the following representation:

$$V(t, x) = V(s, x) + \int_s^t q^0(r, x) dr + \int_s^t \langle\, q(r, x), dW(r)\,\rangle, \qquad (t, x) \in [s, T] \times \mathbb{R}^n,$$

with $q^0(\cdot)$ and $q(\cdot)$ being undetermined. Then by Itô–Kunita's formula, for any $t \in \mathcal{S}[s, T]$,

$$\begin{aligned} V(t, X(t; s, \xi)) = V(s, \xi) + \int_s^t \Big\{ & q^0(r, X(r; s, \xi)) + \langle\, b(r, X(r; s, \xi)), V_x(r, X(r; s, \xi))\,\rangle \\ & + \frac{1}{2}\mathrm{tr}\Big[\sigma(r, X(r; s, \xi))\sigma(r, X(r; s, \xi))^T V_{xx}(r, X(r; s, \xi))\Big] \\ & + \mathrm{tr}\Big[\sigma(r, X(r; s, \xi)) q_x(r, X(r; s, \xi))\Big]\Big\} dr \\ & + \int_s^t \langle\, q(r, X(r; s, \xi)) + \sigma(r, X(r; s, \xi))^T V_x(r, X(r; s, \xi)), dW(r)\,\rangle\,. \end{aligned} \tag{3.12}$$

Hence, by (3.2), we have

$$
\begin{aligned}
0 &\le \mathbb{E}\Big[\int_s^t g(r,X(r;s,\xi))dr + V(t,X(t;s,\xi)) - V(s,\xi)|\mathcal{F}_s\Big] \\
&= \mathbb{E}\Big[\int_s^t \Big\{g(r,X(r;s,\xi)) + q^0(r,X(r;s,\xi)) + \langle\, b(r,X(r;s,\xi)), V_x(r,X(r;s,\xi))\,\rangle \\
&\qquad +\frac{1}{2}\mathrm{tr}\,\Big[\sigma(r,X(r;s,\xi))\sigma(r,X(r;s,\xi))^T V_{xx}(r,X(r;s,\xi))\Big] \\
&\qquad +\mathrm{tr}\,\Big[\sigma(r,X(r;s,\xi))q_x(r,X(r;s,\xi))\Big]\Big\}dr \;\Big|\; \mathcal{F}_s\Big].
\end{aligned}
\tag{3.13}
$$

Dividing it by $(t-s)$ and sending $t \to s$, we obtain

$$
\begin{aligned}
0 \le g(s,\xi) + q^0(s,\xi) + V_x(s,\xi)b(s,\xi) + \frac{1}{2}\mathrm{tr}\,\Big[\sigma(s,\xi)\sigma(s,\xi)^T V_{xx}(s,\xi)\Big] + \mathrm{tr}\,\Big[\sigma(s,\xi)q_x(s,\xi)\Big], \\
\text{a.s. }, \quad \forall (s,\xi) \in \mathcal{D}^p.
\end{aligned}
\tag{3.14}
$$

On the other hand, on the set $\{V(s,\xi) < h(s,\xi)\}$, one has $\bar\tau(s,\xi) > s$, and

$$
\theta \mapsto V(\theta, X(\theta;s,\xi)) + \int_s^\theta g(t,X(t;s,\xi))dt
$$

is a martingale on $[s,\bar\tau(s,\xi))$. Hence, it is necessary that

$$
\begin{aligned}
0 = g(s,\xi) + q^0(s,\xi) + \langle\, b(s,\xi), V_x(s,\xi)^T\,\rangle + \frac{1}{2}\mathrm{tr}\Big[\sigma(s,\xi)\sigma(s,\xi)^T V_{xx}(s,\xi)\Big] + \mathrm{tr}\Big[\sigma(s,\xi)q_x(s,\xi)\Big], \\
\text{a.s. on } \{V(s,\xi) < h(s,\xi)\}, \quad \forall (s,\xi) \in \mathcal{D}^p.
\end{aligned}
\tag{3.15}
$$

Therefore, it is reasonable to require that

$$
\left\{
\begin{aligned}
& q^0(r,x) \ge -\frac{1}{2}\mathrm{tr}\,\Big[\sigma(r,x)\sigma(r,x)^T V_{xx}(r,x)\Big] - \langle\, b(r,x), V_x(r,x)\,\rangle \\
&\qquad -\mathrm{tr}\,\Big[\sigma(r,x)q_x(r,x)\Big] - g(r,x), \qquad \text{a.s. }, \quad (r,x) \in [0,T]\times\mathbb{R}^n, \\
& q^0(r,x) = -\frac{1}{2}\mathrm{tr}\,\Big[\sigma(r,x)\sigma(r,x)^T V_{xx}(r,x)\Big] - \langle\, b(r,x), V_x(r,x)\,\rangle \\
&\qquad -\mathrm{tr}\Big[\sigma(r,x)q_x(r,x)\Big] - g(r,x), \quad \text{a.s. on } \{V(r,x) < h(r,x)\}, \; (r,x) \in [0,T]\times\mathbb{R}^n.
\end{aligned}
\right.
\tag{3.16}
$$

If we let $\beta : \mathbb{R} \to [0,+\infty]$ be a monotone graph defined by

$$
\beta(\rho) = \left\{
\begin{array}{ll}
[0,+\infty], & \rho \ge 0, \\
0, & \rho < 0,
\end{array}
\right.
\tag{3.17}
$$

then we should have

$$
\begin{aligned}
q^0(r,x) \in -\frac{1}{2}\mathrm{tr}\,\Big[\sigma(r,x)\sigma(r,x)^T V_{xx}(r,x)\Big] - \langle\, b(r,x), V_x(r,x)\,\rangle - \mathrm{tr}\,\Big[\sigma(r,x)q_x(r,x)\Big] \\
+\beta\Big(V(r,x) - h(r,x)\Big) - g(r,x), \qquad \text{a.s. }, \quad (r,x) \in [0,T]\times\mathbb{R}^n,
\end{aligned}
\tag{3.18}
$$

which is understood as follows:

$$
\left\{
\begin{aligned}
& q^0(r,x) = -\frac{1}{2}\mathrm{tr}\,\Big[\sigma(r,x)\sigma(r,x)^T V_{xx}(r,x)\Big] - \langle\, b(r,x), V_x(r,x)\,\rangle - \mathrm{tr}\,\Big[\sigma(r,x)q_x(r,x)\Big] \\
&\qquad +\zeta(r,x) - g(r,x), \qquad \text{a.s. }, \quad (r,x) \in [0,T]\times\mathbb{R}^n, \\
& \zeta(r,x) \in \beta\Big(V(r,x) - h(r,x)\Big), \qquad \text{a.s. }, \quad (r,x) \in [0,T]\times\mathbb{R}^n,
\end{aligned}
\right.
\tag{3.19}
$$

In the above, $\zeta(\cdot,x)$ is required to be $\mathbb{F}$-adapted. Consequently, we should have

$$\begin{cases} V(t,x)=h(T,x)+\displaystyle\int_t^T\Big\{\frac{1}{2}\mathrm{tr}\Big[\sigma(r,x)\sigma(r,x)^TV_{xx}(r,x)\Big]+\langle\, b(r,x),V_x(r,x)\rangle+\mathrm{tr}\Big[\sigma(r,x)q_x(r,x)\Big]\\ \qquad\qquad -\zeta(r,x)+g(r,x)\Big\}dr-\displaystyle\int_t^T\langle\, q(r,x),dW(r)\,\rangle, \qquad t\in[0,T],\quad x\in\mathbb{R}^n,\\ \zeta(t,x)\in\beta\Big(V(t,x)-h(t,x)\Big), \qquad t\in[0,T],\quad x\in\mathbb{R}^n. \end{cases} \tag{3.20}$$

We call (3.20) a *backward stochastic partial differential variational inequality* (BSPDVI, for short). Note that in (3.20), the unknown is the triple of $\mathbb{F}$-adapted random fields $(V,q,\zeta):[0,T]\times\mathbb{R}^n\times\Omega\to\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$. Note that last inclusion in (3.20) is equivalent to the following:

$$\begin{cases} V(t,x)-h(t,x)\le 0,\quad \zeta(t,x)\ge 0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s. },\\ \Big[V(t,x)-h(t,x)\Big]\zeta(t,x)=0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s. }, \end{cases} \tag{3.21}$$

4 Adapted Solutions.

In this section, we will introduce notions of adapted solutions for BSPDVI (3.20), and will carry out some preliminary studies. To begin with, let us make a little preparations.

By a multi-index α we mean $\alpha=(\alpha_1,\cdots,\alpha_n)$ with each α_i being nonnegative integers, and we define $|\alpha|=\sum_{i=1}^n\alpha_i$. We write $x=(x_1,\cdots,x_n)$ for any generic point in $\mathbb{R}^n$. For any multi-index $\alpha\equiv(\alpha_1,\cdots,\alpha_n)$ and any smooth function $f(\cdot)$, denote

$$\partial^\alpha f(x)=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}f(x). \tag{4.1}$$

For any domain $G\subseteq\mathbb{R}^n$ (G is allowed to be $\mathbb{R}^n$), let $C^k(G,\mathbb{R})$ be the set of all functions $f:G\to\mathbb{R}$ such that

$$\sup_{x\in G,\,|\alpha|\le k}|\partial^\alpha f(x)|<\infty. \tag{4.2}$$

We may similarly define the spaces $C^k(G;\mathbb{R}^n)$ and $C^k(G;\mathbb{R}^{n\times d})$, etc. Clearly, these are Banach spaces. Next, we let $W^{m,p}(G;\mathbb{R})$ be the usual Sobolev space of all functions $f(\cdot)$ such that

$$\|f(\cdot)\|^p_{W^{m,p}(G)}\equiv\sum_{|\alpha|\le m}\int_G|\partial^\alpha f(x)|^pdx<\infty, \tag{4.3}$$

and $H^m(\mathbb{R}^n)=W^{m,2}(\mathbb{R}^n)$. For any Banach space B, let $L^\infty_{\mathbb{F}}(0,T;B)$ be the space of all bounded $\mathbb{F}$-progressively measurable maps $f:[0,T]\times\Omega\to B$, with the norm

$$\|f(\cdot)\|_{L^\infty_{\mathbb{F}}(0,T;B)}=\operatorname*{esssup}_{(t,\omega)\in[0,T]\times\Omega}\|f(t,\omega)\|_B. \tag{4.4}$$

Here, B could be $C^k(G;\mathbb{R}^n)$, say. Similarly, we let $C_{\mathbb{F}}(0,T;B)$ be the space of all B-valued $\mathbb{F}$-adapted continuous processes, which is a closed subspace of $L^\infty_{\mathbb{F}}(0,T;B)$.

We now introduce the following definition.

Definition 4.1. (i) *A triple of random fields* (V,q,ζ) *is called an* adapted strong solution *of* (3.20) *if for each* $x\in\mathbb{R}^n$, $t\mapsto(V(t,x),q(t,x),\zeta(t,x))$ *is* $\mathbb{F}$*-adapted and for almost all* $(t,x,\omega)\in[0,T]\times\mathbb{R}^n\times\Omega$, $x\mapsto V(t,x,\omega)$ *is twice differentiable,* $x\mapsto q(t,x,\omega)$ *is once differentiable such that* (3.20) *is satisfied for almost all* $(x,\omega)\in\mathbb{R}^n\times\Omega$.

(ii) *An adapted strong solution* (V,q,ζ) *of* (3.20) *is called an* adapted classical solution *of* (3.20) *if for almost all* $(t,\omega)\in[0,T]\times\Omega$, $x\mapsto V(t,x,\omega)$ *is* C^2, $x\mapsto q(t,x,\omega)$ *is* C^1, *and* $x\mapsto\zeta(t,x)$ *is continuous.*

Once we have the well-posedness of our BSPDVI (which will be treated in next section), it is natural to ask if the solution $V(\cdot\,,\cdot)$ to the BSPDVI has anything to do with our Problem (S)? The following result answers this question: Under appropriate conditions, the solution of BSPDVI (3.20) coincides with value function of Problem (S), via which an optimal stopping time can be identified.

Theorem 4.2. *Let* (H1)–(H2) *hold. Suppose* (V, q, ζ) *is an adapted classical solution to BSPDVI* (3.20). *Then* $V(\cdot\,,\cdot)$ *is the value function of Problem (S). Consequently the part* $V(\cdot\,,\cdot)$ *of the adapted classical solution* (V, q, ζ) *to* (3.20) *is unique. Moreover, the following gives the smallest optimal stopping time of Problem (S):*

$$\bar\tau(s,\xi) = \inf\Big\{t \in [s,T] \bigm| V(t, X(t; s,\xi)) = h(t, X(t; s,\xi))\Big\}. \tag{4.5}$$

Proof. Let $(s,\xi) \in \mathcal{D}^p$, and define $\bar\tau(s,\xi)$ by (4.5). By Itô–Kunita Formula, together with the BSPDVI (3.20), we have

$$V(s,\xi) = \mathbb{E}\Big[\int_s^\tau g(r, X(r; s,\xi))dr + V(\tau, X(\tau; s,\xi))\big|\mathcal{F}_s\Big], \quad \forall \tau \in \mathcal{S}[s, \bar\tau(s,\xi)]. \tag{4.6}$$

Hence, taking $\tau = \bar\tau(s,\xi)$, we have

$$\begin{aligned} V(s,\xi) &= \mathbb{E}\Big[V(\bar\tau(s,\xi), X(\bar\tau(s,\xi); s,\xi)) + \int_s^{\bar\tau(s,\xi)} g(r, X(r; s,\xi))dr \bigm| \mathcal{F}_s\Big] \\ &= \mathbb{E}\Big[h(\bar\tau(s,\xi), X(\bar\tau(s,\xi); s,\xi)) + \int_s^{\bar\tau(s,\xi)} g(r, X(r; s,\xi))dr \bigm| \mathcal{F}_s\Big] = J_{s,\xi}(\bar\tau(s,\xi)). \end{aligned} \tag{4.7}$$

This means that $\bar\tau(s,\xi)$ is an optimal stopping time for our Problem (S). From the above, we further conclude that the part $V(\cdot\,,\cdot)$ of the adapted solution (V, q, ζ) to BSPDVI (3.20) is unique, and from (4.5), $\bar\tau(s,\xi)$ has to be the smallest optimal stopping time (noting (ii) of Theorem 3.1). □

Next, we would like to make a reduction which will be very useful below. To this end, let

$$h(t,x) = h(0,x) + \int_0^t \mu^0(r,x)dr + \int_0^t \langle\, \mu(r,x), dW(r)\,\rangle, \qquad t \in [0,T], \tag{4.8}$$

for some suitable $\mu^0(\cdot)$ and $\mu(\cdot)$. Suppose (V, q, ζ) is an adapted classical solution to the BSPDVI (3.20), and all the coefficients have required order of derivatives. We fix a $p \geq 2$ and let

$$\begin{cases} \bar V(t,x) = \dfrac{V(t,x) - h(t,x)}{1 + |x|^p}\,, \\ \bar q(t,x) = \dfrac{q(t,x) - \mu(t,x)}{1 + |x|^p}\,, & (t,x) \in [0,T] \times \mathbb{R}^n, \text{ a.s.} \\ \bar\zeta(t,x) = \dfrac{\zeta(t,x)}{1 + |x|^p}\,, \end{cases} \tag{4.9}$$

Note that in the case $x \mapsto V(t,x) - h(t,x)$ grows at most linearly, $x \mapsto \bar V(t,x)$ will be L^2-integrable over $\mathbb{R}^n$. According to the above, one has

$$\begin{cases} V(t,x) = (1 + |x|^p)\bar V(t,x) + h(t,x), \\ q(t,x) = (1 + |x|^p)\bar q(t,x) + \mu(t,x), \\ \zeta(t,x) = (1 + |x|^p)\bar\zeta(t,x). \end{cases} \tag{4.10}$$

Consequently (suppressing (t,x))

$$\begin{cases} V_x = (1 + |x|^p)\bar V_x + p|x|^{p-2}\bar V x + h_x, \\ V_{xx} = (1 + |x|^p)\bar V_{xx} + p|x|^{p-2}\Big[(x\bar V_x)^T + x\bar V_x\Big] + \Big[p(p-2)|x|^{p-4}xx^T + p|x|^{p-2}I\Big]\bar V + h_{xx}, \\ q_x = (1 + |x|^p)\bar q_x + p|x|^{p-2}\bar q x^T + \mu_x. \end{cases} \tag{4.11}$$

Hence,

$$
\begin{aligned}
&\bar V(t,x)\equiv(1+|x|^p)^{-1}[V(t,x)-h(t,x)]\\
&=(1+|x|^p)^{-1}\Big(h(T,x)-h(t,x)+\int_t^T\Big\{\frac12\mathrm{tr}\Big[\sigma(r,x)\sigma(r,x)^TV_{xx}(r,x)\Big]+\langle\, b(r,x),V_x(r,x)\,\rangle\\
&\qquad+\mathrm{tr}\Big[\sigma(r,x)q_x(r,x)\Big]-\zeta(r,x)+g(r,x)\Big\}dr-\int_t^T\langle\, q(r,x),dW(r)\,\rangle\Big)\\
&=\int_t^T\frac{\mu^0(r,x)}{1+|x|^p}dr+\int_t^T\langle\,\frac{\mu(r,x)}{1+|x|^p},dW(r)\,\rangle+\int_t^T\Big\{\frac12\mathrm{tr}\Big[\sigma(r,x)\sigma(r,x)^T\\
&\qquad\cdot\Big(\bar V_{xx}+\frac{p|x|^{p-2}[x\bar V_x^T+\bar V_xx^T]}{1+|x|^p}+\frac{p|x|^{p-4}[(p-2)xx^T+|x|^2I]\bar V+h_{xx}}{1+|x|^p}\Big)\Big]\\
&\qquad+\langle\, b(r,x),\bar V_x+\frac{p|x|^{p-2}\bar Vx+h_x}{1+|x|^p}\,\rangle+\mathrm{tr}\Big[\sigma(r,x)\Big(\bar q_x+\frac{p|x|^{p-2}\bar qx^T+\mu_x}{1+|x|^p}\Big)\Big]\\
&\qquad-\frac{\zeta(r,x)-g(r,x)}{1+|x|^p}\Big\}dr-\int_t^T\langle\,\frac{q(r,x)}{1+|x|^p},dW(s)\,\rangle\\
&=\int_t^T\Big\{\frac12\mathrm{tr}\Big[\sigma(r,x)\sigma(r,x)^T\bar V_{xx}\Big]+\langle\, b(r,x)+\frac{p|x|^{p-2}\sigma(r,x)\sigma(r,x)^Tx}{1+|x|^p},\bar V_x\,\rangle\\
&\qquad+\frac{p|x|^{p-4}[(p-2)|\sigma(r,x)^Tx|^2+|x|^2|\sigma(r,x)|^2]+2p|x|^{p-2}\langle\, b(r,x),x\,\rangle}{2(1+|x|^p)}\bar V\\
&\qquad+\mathrm{tr}\Big[\sigma(r,x)\bar q_x\Big]+\langle\,\frac{p|x|^{p-2}\sigma(r,x)^Tx}{1+|x|^p},\bar q\,\rangle-\bar\zeta\\
&\qquad+(1+|x|^p)^{-1}\Big[\frac12\mathrm{tr}\Big(\sigma(r,x)\sigma(r,x)^Th_{xx}(r,x)\Big)+\langle\, b(r,x),h_x(r,x)\,\rangle\\
&\qquad+\mathrm{tr}\Big(\sigma(r,x)\mu_x(r,x)\Big)+\mu^0(r,x)+g(r,x)\Big]\Big\}dr-\int_s^t\langle\,\bar q(r,x),dW(s)\,\rangle\\
&\equiv\int_s^t\Big\{\frac12\mathrm{tr}\Big[\sigma(r,x)\sigma(r,x)^T\bar V_{xx}(r,x)\Big]+\langle\,\widetilde b(r,x),\bar V_x(r,x)\,\rangle+\widetilde b^0(r,x)\bar V(r,x)\\
&\qquad+\mathrm{tr}\Big[\sigma(r,x)\bar q_x(r,x)\Big]+\langle\,\widetilde\sigma^0(r,x),\bar q(r,x)\,\rangle-\bar\zeta(r,x)+\widetilde g(r,x)\Big\}dr-\int_s^t\langle\,\bar q(r,x),dW(s)\,\rangle,
\end{aligned}
\tag{4.12}
$$

with

$$
\begin{cases}
\widetilde b(r,x)=b(r,x)+\dfrac{p|x|^{p-2}\sigma(r,x)\sigma(r,x)^Tx}{1+|x|^p},\\
\widetilde b^0(r,x)=\dfrac{p|x|^{p-4}[(p-2)|\sigma(r,x)^Tx|^2+|x|^2|\sigma(r,x)|^2]+2p|x|^{p-2}\langle\, b(r,x),x\,\rangle}{2(1+|x|^p)},\\
\widetilde\sigma^0(r,x)=\dfrac{p|x|^{p-2}\sigma(r,x)^Tx}{1+|x|^p},\\
\widetilde g(r,x)=(1+|x|^p)^{-1}\Big[\dfrac12\mathrm{tr}\Big(\sigma(r,x)\sigma(r,x)^Th_{xx}(r,x)\Big)+\langle\, b(r,x),h_x(r,x)\,\rangle\\
\qquad\qquad+\mathrm{tr}\Big(\sigma(r,x)\mu_x(r,x)\Big)+\mu^0(r,x)+g(r,x)\Big].
\end{cases}
\tag{4.13}
$$

Note that by the definition of β, we see that

$$\zeta(t,x)\in\beta\Big(V(t,x)-h(t,x)\Big),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\quad \text{a.s.}\tag{4.14}$$

is equivalent to

$$\bar\zeta(t,x)\in\beta\Big(\bar V(t,x)\Big),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\quad \text{a.s.}\tag{4.15}$$

Next, we have

$$
\begin{cases}
\mathrm{tr}\Big(\sigma\sigma^T\bar V_{xx}\Big)=\nabla\cdot(\sigma\sigma^T\bar V_x)-\langle\,\nabla\cdot(\sigma\sigma^T),\bar V_x\,\rangle\\
\mathrm{tr}\,(\sigma\bar q_x)=\nabla\cdot(\sigma\bar q)-\langle\,\nabla\cdot\sigma,\bar q\,\rangle.
\end{cases}
\tag{4.16}
$$

where (with $\sigma=(\sigma_1,\cdots,\sigma_d)$, each σ_i takes values in $\mathbb{R}^n$)

$$\nabla\cdot\sigma=(\nabla\cdot\sigma_1\,,\cdots,\nabla\cdot\sigma_d)^T, \tag{4.17}$$

and

$$\begin{aligned}\nabla\cdot(\sigma\sigma^T)&=\sum_{k=1}^d\nabla\cdot(\sigma_k\sigma_k^T)=\sum_{k=1}^d\Big(\nabla\cdot(\sigma_{1k}\sigma_k),\cdots,\nabla\cdot(\sigma_{nk}\sigma_k)\Big)^T\\&=\sum_{k=1}^d\Big(\sigma_{1k}(\nabla\cdot\sigma_k),\cdots,\sigma_{nk}(\nabla\cdot\sigma_k)\Big)^T+\sum_{k=1}^d\Big(\sigma_k^T(\sigma_{1k})_x,\cdots,\sigma_k^T(\sigma_{nk})_x\Big)^T\\&=\sum_{k=1}^d(\nabla\cdot\sigma_k)\sigma_k+\sum_{k=1}^d(\sigma_k)_x\sigma_k=\sum_{k=1}^d\Big[(\nabla\cdot\sigma_k)I+(\sigma_k)_x\Big]\sigma_k.\end{aligned} \tag{4.18}$$

Then, we can get

$$\begin{aligned}&\frac12\mathrm{tr}\Big(\sigma\sigma^T\bar V_{xx}\Big)+\langle\,\widetilde b,\bar V_x\,\rangle+\widetilde b^0\bar V+\mathrm{tr}\Big(\sigma\bar q_x\Big)+\langle\,\widetilde\sigma^0,\bar q\,\rangle-\bar\zeta+\widetilde g\\&=\frac12\nabla\cdot(\sigma\sigma^T\bar V_x)+\langle\,\widetilde b-\nabla\cdot(\sigma\sigma^T),\bar V_x\,\rangle+\widetilde b^0\bar V+\nabla\cdot(\sigma\bar q)+\langle\,\widetilde\sigma^0-\nabla\cdot\sigma,\bar q\,\rangle-\bar\zeta+\bar g.\end{aligned} \tag{4.19}$$

According to the above reduction, we have the following divergence form of our BSPDVI:

$$\begin{cases}\bar V(t,x)=\displaystyle\int_t^T\Big\{\frac12\nabla\cdot\Big[\sigma(r,x)\sigma(r,x)^T\bar V_x(r,x)\Big]+\langle\,\bar b(r,x),\bar V_x(r,x)\,\rangle+\widetilde b^0(r,x)\bar V(r,x)\\\qquad\qquad+\nabla\cdot\Big[\sigma(r,x)\bar q(r,x)\Big]+\langle\,\bar\sigma^0(r,x),\bar q(r,x)\,\rangle-\bar\zeta(r,x)+\bar g(r,x)\Big\}dr\\\qquad\qquad-\displaystyle\int_t^T\langle\,\bar q(r,x),dW(r)\,\rangle,\qquad t\in[0,T],\quad x\in\mathbb{R}^n,\\\bar\zeta(t,x)\in\beta\Big(\bar V(t,x)\Big),\qquad t\in[0,T],\quad x\in\mathbb{R}^n,\end{cases} \tag{4.20}$$

with

$$\begin{cases}\bar b(r,x)=\widetilde b(r,x)-\nabla\cdot\Big[\sigma(r,x)\sigma(r,x)^T\Big]\\\qquad=b(r,x)-\nabla\cdot\Big[\sigma(r,x)\sigma(r,x)^T\Big]+\dfrac{p|x|^{p-2}\sigma(r,x)\sigma(r,x)^Tx}{1+|x|^p},\\\bar b^0(r,x)=\widetilde b^0(r,x)=\dfrac{p|x|^{p-4}[(p-2)|\sigma(r,x)^Tx|^2+|x|^2|\sigma(r,x)|^2]+2p|x|^{p-2}\,\langle\,b(r,x),x\,\rangle}{2(1+|x|^p)},\\\bar\sigma^0(r,x)=\widetilde\sigma^0(r,x)-\nabla\cdot\sigma(r,x)=\dfrac{p|x|^{p-2}\sigma(r,x)^Tx}{1+|x|^p}-\nabla\cdot\sigma(r,x)\,,\\\bar g(r,x)=\widetilde g(r,x)=(1+|x|^p)^{-1}\Big[\dfrac12\mathrm{tr}\Big(\sigma(r,x)\sigma(r,x)^Th_{xx}(r,x)\Big)+\langle\,b(r,x),h_x(r,x)\,\rangle\\\qquad+\mathrm{tr}\Big(\sigma(r,x)\mu_x(r,x)\Big)+\mu^0(r,x)+g(r,x)\Big].\end{cases} \tag{4.21}$$

In order the above reduction to be possible, and for the purpose of some other further discussions, we introduce the following assumption.

(H3) *For some $k>2+\frac n2$, the maps $b(\cdot)$, $\sigma(\cdot)$, $g(\cdot)$, $\mu_0(\cdot)$, $\mu(\cdot)$, and $h(0,\cdot)$ satisfy the following:*

$$\begin{cases}b(\cdot)\in L^\infty_{\mathbb F}(0,T;C^k(\mathbb{R}^n;\mathbb{R}^n)),\quad\sigma(\cdot),\mu(\cdot)\in L^\infty_{\mathbb F}(0,T;C^k(\mathbb{R}^n;\mathbb{R}^{n\times d})),\\g(\cdot),\mu_0(\cdot)\in L^\infty_{\mathbb F}(0,T;C^k(\mathbb{R}^n;\mathbb{R})),\quad h(0,\cdot)\in C^k(\mathbb{R}^n;\mathbb{R}).\end{cases} \tag{4.22}$$

Under (H3), we see that

$$\begin{cases}|\bar b(t,x)|+|\bar b^0(t,x)|+|\bar\sigma^0(t,x)|\le C,\\|\bar g(t,x)|\le C(1+|x|^p)^{-1},\end{cases}\qquad(t,x)\in(t,x)\in[0,T]\times\mathbb{R}^n,\quad\text{a.s.} \tag{4.23}$$

In what follows, we will choose $p>2$ which will lead to

$$\int_{\mathbb{R}^n} |\bar g(t,x)|dx \le C, \qquad \forall t\in[0,T], \text{ a.s.} \tag{4.24}$$

We may introduce adapted classical and strong solutions to the divergence form of BSPDVI (4.20) similar to Definition 4.1. On the other hand, let us now introduce the following notion.

Definition 4.3. *A triple*

$$(\bar V,\bar q,\bar\zeta)\in L^2_{\mathbb{F}}(0,T;H^1(\mathbb{R}^n))\times L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n;\mathbb{R}^d))\times L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n)) \tag{4.25}$$

is called an adapted weak solution of (4.20) if for any $\varphi\in H^1(\mathbb{R}^n)$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\bar V(t,x)\varphi(x)dx=\int_t^T\int_{\mathbb{R}^n}\Big\{-\langle\,\frac12\sigma(r,x)^T\bar V_x(r,x)+\bar q(r,x),\sigma(r,x)^T\varphi_x(x)\,\rangle\\&\quad+\Big[\,\langle\,\bar b(r,x),\bar V_x(r,x)\,\rangle+\bar b^0(r,x)\bar V(r,x)+\langle\,\bar\sigma^0(r,x),\bar q(r,x)\,\rangle+\bar g(r,x)-\bar\zeta(r,x)\Big]\varphi(x)\Big\}dxdr\\&\quad-\int_t^T\langle\int_{\mathbb{R}^n}\bar q(r,x)\varphi(x)dx,dW(r)\,\rangle,\qquad t\in[0,T].\end{aligned} \tag{4.26}$$

and

$$\begin{cases}\bar V(t,x)\le0,\quad \bar\zeta(t,x)\ge0, & (t,x)\in[0,T]\times\mathbb{R}^n,\quad \text{a.s. },\\ \bar V(t,x)\bar\zeta(t,x)=0, & \text{a.e. } (t,x)\in[0,T]\times\mathbb{R}^n,\quad \text{a.s.}\end{cases} \tag{4.27}$$

We point out that any adapted strong solution $(\bar V,\bar q,\bar\zeta)$ of (4.20) must be an adapted weak solution of (4.20). Similar to [18], we can show, by an argument using integration by parts, that an adapted weak solution is an adapted strong (classical) solution if it have the regularity that the later requires.

5 Well-Posedness of the BSPDVI

In this section, we are going to discuss the issue of the well-posedness of BSPVDI. First, we have the following.

Theorem 5.1. *Suppose* (H3) *holds. Let $(\bar V,\bar q,\bar\zeta)$ and $(\widetilde V,\widetilde q,\widetilde\zeta)$ be adapted weak solutions to* (4.20) *with*

$$\begin{cases}\bar V,\widetilde V\in L^2_{\mathbb{F}}(0,T;H^1(\mathbb{R}^n)),\\ \bar q,\widetilde q\in L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n;\mathbb{R}^d)),\\ \bar\zeta,\widetilde\zeta\in L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n)).\end{cases} \tag{5.1}$$

Then

$$\bar V(t,x)=\widetilde V(t,x),\quad \bar q(t,x)=\widetilde q(t,x),\quad \bar\zeta(t,x)=\widetilde\zeta(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s.} \tag{5.2}$$

Proof. Suppose $(\bar V,\bar q,\bar\zeta)$ and $(\widetilde V,\widetilde q,\widetilde\zeta)$ are two adapted weak solutions to BSPDVI (4.20). Set

$$\widehat V=\bar V-\widetilde V,\qquad \widehat q=\bar q-\widetilde q,\qquad \widehat\zeta=\bar\zeta-\widetilde\zeta. \tag{5.3}$$

Then $(\widehat V,\widehat q,\widehat\zeta)$ is an adapted weak solution to the following linear BSPDE:

$$\begin{cases}\widehat V(t,x)=\displaystyle\int_t^T\Big\{\frac12\nabla\cdot\Big[\sigma(r,x)\sigma(r,x)^T\widehat V_x(r,x)\Big]+\langle\,\bar b(r,x),\widehat V_x(r,x)\,\rangle+\bar b^0(r,x)\widehat V(r,x)\\ \qquad\qquad+\nabla\cdot\Big[\sigma(r,x)\widehat q(r,x)\Big]+\langle\,\bar\sigma^0(r,x),\widehat q(r,x)\,\rangle-\widehat\zeta(r,x)\Big\}dr\\ \qquad\qquad-\displaystyle\int_t^T\langle\,\widehat q(r,x),dW(r)\,\rangle,\qquad t\in[0,T],\quad x\in\mathbb{R}^n.\end{cases} \tag{5.4}$$

By an Itô's type formula (see [18] for details), we have

$$
\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^n}|\widehat V(t,x)|^2dx &= \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{-|\sigma^T\widehat V_x|^2+[2\bar b^0-(\nabla\cdot\bar b)]\widehat V^2-2\langle\widehat q,\sigma^T\widehat V_x\rangle \\
&\qquad +2\langle\bar\sigma^0,\widehat q\rangle\widehat V-2\widehat\zeta\widehat V-|\widehat q|^2\Big\}dxdr \\
&= \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{-|\sigma^T\widehat V_x+\widehat q-\bar\sigma^0\widehat V|^2+|\bar\sigma^0|^2\widehat V^2-2\langle\sigma^T\widehat V_x,\bar\sigma^0\widehat V\rangle \\
&\qquad +[2\bar b^0-(\nabla\cdot\bar b)]\widehat V^2-2\widehat\zeta\widehat V\Big\}dxdr \\
&= \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{-|\sigma^T\widehat V_x+\widehat q-\bar\sigma^0\widehat V|^2+\Big[|\bar\sigma^0|^2+\nabla\cdot(\sigma\bar\sigma^0)+2\bar b^0-(\nabla\cdot\bar b)\Big]\widehat V^2-2\widehat\zeta\widehat V\Big\}dxdr
\end{aligned}
\tag{5.5}
$$

Note that

$$
\widehat V(t,x)\widehat\zeta(t,x)\equiv\Big[\bar V(t,x)-\widetilde V(t,x)\Big]\Big[\bar\zeta(t,x)-\widetilde\zeta(t,x)\Big]=-\bar V(t,x)\widetilde\zeta(t,x)-\widetilde V(t,x)\bar\zeta(t,x)\geq 0. \tag{5.6}
$$

Thus, the above implies that

$$
\mathbb{E}\int_{\mathbb{R}^n}|\bar V(t,x)-\widetilde V(t,x)|^2dx\leq C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}|\bar V(r,x)-\widetilde V(r,x)|^2dxdr. \tag{5.7}
$$

Hence, by Gronwall's inequality, we obtain that

$$
\bar V(t,x)=\widetilde V(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s.} \tag{5.8}
$$

Further, (5.5) implies that

$$
\bar q(t,x)=\widetilde q(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s.} \tag{5.9}
$$

Then, by virtue of (5.4), we have

$$
\bar\zeta(t,x)=\widetilde\zeta(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s.} \tag{5.10}
$$

which proves our conclusion. □

To establish the existence of an adapted weak solution, we define

$$
\eta(\rho)=\begin{cases}0, & \rho\in(-\infty,0]\bigcup(2,\infty),\\ \rho, & \rho\in(0,1],\\ 2-\rho, & \rho\in(1,2],\end{cases} \tag{5.11}
$$

and define

$$
\psi(\rho)=\int_0^\rho\int_0^\tau\eta(r)drd\tau=\int_0^\rho(\rho-r)\eta(r)dr,\qquad \rho\in\mathbb{R}. \tag{5.12}
$$

Thus, $\psi:\mathbb{R}\to\mathbb{R}$ is C^2, nondecreasing, and convex.

Now, for any $\varepsilon>0$, we consider the following semilinear backward stochastic partial differential equation (BSPDE, for short):

$$
\begin{aligned}
V^\varepsilon(t,x)&=\int_t^T\Big\{\frac12\nabla\cdot\Big(\sigma\sigma^TV_x^\varepsilon\Big)+\langle\bar b,V_x^\varepsilon\rangle+\bar b^0V^\varepsilon+\nabla\cdot(\sigma q^\varepsilon)+\langle\bar\sigma^0,q^\varepsilon\rangle\\
&\qquad+\bar g-\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big\}dr-\int_t^T\langle q^\varepsilon,dW(r)\rangle,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,
\end{aligned}
\tag{5.13}
$$

The unknown of the semilinear BSPDE (5.13) is the pair $(V^\varepsilon, q^\varepsilon)$ of $\mathbb{F}$-adapted random fields. The following is a special case of a relevant result found in [28].

Theorem 5.2. *Let* (H3) *hold. Then the semilinear BSPDE* (5.13) *admits a unique adapted classical solution* $(V^\varepsilon, q^\varepsilon)$*. Moreover, for any* $p > 1$ *and any compact set* $K \subseteq \mathbb{R}^n$,

$$\mathbb{E}\Big[\sup_{t\in[0,T],\, x\in K} |\partial^\alpha V^\varepsilon(t,x)|^p\Big] + \mathbb{E}\int_0^T \sup_{x\in K} |\partial^\alpha q^\varepsilon(t,x)|^p dt < \infty, \quad \forall |\alpha| \le 2. \tag{5.14}$$

We hope that the unique adapted classical solution $(V^\varepsilon, q^\varepsilon)$ of (4.20) will converge to $(\bar V, \bar q)$ in some sense, where $(\bar V, \bar q, \bar\zeta)$ is the adapted weak solution to our BSPDVI (4.20). Moreover, it is a hope that the value function V of Problem (S) can be identified by $\bar V$ through (4.10). However, we note that in the above estimate (5.14), the bound of the left hand side not only depends on the compact set K, but also depends on $\varepsilon > 0$, in general. Hence, we first would like to establish some estimates for $(V^\varepsilon, q^\varepsilon)$ (on the whole space $[0,T]\times\mathbb{R}^n$) which are uniform in $\varepsilon > 0$. To this end, we begin with several lemmas whose technical proofs will be given in the appendix.

Lemma 5.3. *Let* $\theta : \mathbb{R} \to [0,\infty)$ *be convex and piecewise smooth. Suppose that*

$$0 = \theta(0) = \min_{\rho\in\mathbb{R}} \theta(\rho), \tag{5.15}$$

and

$$[\theta'(\rho)]^2 \le C\theta(\rho)\theta''(\rho), \qquad \text{a.e. } \rho\in\mathbb{R}, \tag{5.16}$$

for some constant $C > 0$*. Let* $(V^\varepsilon, q^\varepsilon)$ *be the adapted classical solution to BSPDE* (5.13)*. Then*

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^n} \theta\Big(V^\varepsilon(t,x)\Big)dx + \mathbb{E}\int_t^T\int_{\mathbb{R}^n} \Big\{\theta''(V^\varepsilon(r,x))\Big|\sigma(r,x)^T V_x^\varepsilon(r,x) + q^\varepsilon(r,x)\Big|^2\\ &+\theta'(V^\varepsilon(r,x))\psi\Big(\frac{V^\varepsilon(r,x)}{\varepsilon}\Big)\Big\}dxdr \le C\mathbb{E}\int_t^T\int_{\mathbb{R}^n} |\theta'(V^\varepsilon(r,x))|\Big(|V^\varepsilon(r,x)| + |\bar g(r,x)|\Big)dxdr,\\ &\qquad\qquad t\in[0,T].\end{aligned} \tag{5.17}$$

The above lemma can be used to establish several interesting estimates for $(V^\varepsilon, q^\varepsilon)$.

Lemma 5.4. *Let* (H3) *hold and* $(V^\varepsilon, q^\varepsilon)$ *be the classical solution to the BSPDE* (5.13)*. Then there exists a constant* $C > 0$*, independent of* $m \ge 1$ *and* $\varepsilon > 0$*, such that*

$$\begin{aligned}&\sup_{t\in[0,T]} \mathbb{E}\int_{\mathbb{R}^n} |V^\varepsilon(t,x)|^{2m}dx + \mathbb{E}\int_0^T\int_{\mathbb{R}^n} m^2|V^\varepsilon(r,x)|^{2m-2}\Big[|\sigma(r,x)^T V_x^\varepsilon(r,x) + q^\varepsilon(r,x)|^2\\ &\quad+mV^\varepsilon(r,x)\psi\Big(\frac{V^\varepsilon(r,x)}{\varepsilon}\Big)\Big]dxdr \le Ce^{Cm}\mathbb{E}\int_0^T\int_{\mathbb{R}^n} |\bar g(r,x)|^{2m}dxdr, \quad \forall\varepsilon>0,\ m\ge1,\end{aligned} \tag{5.18}$$

$$\begin{aligned}&\sup_{t\in[0,T]} \mathbb{E}\int_{\mathbb{R}^n} \Big(V^\varepsilon(t,x)^+\Big)^2 dx + \mathbb{E}\int_0^T\int_{\mathbb{R}^n} \Big\{\Big|\sigma(r,x)^T V_x^\varepsilon(r,x) + q^\varepsilon(r,x)\Big|^2 I_{\{V^\varepsilon>0\}}\\ &\quad+V^\varepsilon(r,x)^+\psi\Big(\frac{V^\varepsilon(r,x)}{\varepsilon}\Big)\Big\}dxdr \le C\mathbb{E}\int_0^T\int_{\mathbb{R}^n} |\bar g(r,x)|^2 I_{\{V^\varepsilon>0\}}dxdr, \qquad \forall\varepsilon>0,\end{aligned} \tag{5.19}$$

$$\sup_{t\in[0,T]} \mathbb{E}\int_{\mathbb{R}^n} V^\varepsilon(t,x)\psi\Big(\frac{V^\varepsilon(t,x)}{\varepsilon}\Big)dx + \mathbb{E}\int_0^T\int_{\mathbb{R}^n} \psi\Big(\frac{V^\varepsilon(r,x)}{\varepsilon}\Big)^2 dxdr \le C\mathbb{E}\int_0^T\int_{\mathbb{R}^n} |\bar g(r,x)|^2 dxdr. \tag{5.20}$$

Next, differentiating the BSPDE (5.13) with respect to x_k, we get

$$\begin{aligned}V^\varepsilon_{x_k}(t,x)=\int_t^T&\Big\{\frac{1}{2}\nabla\cdot\Big(\sigma\sigma^T(V^\varepsilon_{x_k})_x\Big)+\langle\,\bar b,(V^\varepsilon_{x_k})_x\,\rangle+\bar b^0V^\varepsilon_{x_k}+\nabla\cdot(\sigma q^\varepsilon_{x_k})+\langle\,\bar\sigma^0,q^\varepsilon_{x_k}\,\rangle\\&-\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{V^\varepsilon_{x_k}}{\varepsilon}+\frac{1}{2}\nabla\cdot\Big((\sigma\sigma^T)_{x_k}V^\varepsilon_x\Big)+\langle\,\bar b_{x_k},V^\varepsilon_x\,\rangle+\bar b^0_{x_k}V^\varepsilon\\&+\nabla\cdot(\sigma_{x_k}q^\varepsilon)+\langle\,\bar\sigma^0_{x_k},q^\varepsilon\,\rangle+\bar g_{x_k}\Big\}dr-\int_t^T\langle\,q^\varepsilon_{x_k},dW(r)\,\rangle\,.\end{aligned}\tag{5.21}$$

We have the following result.

Lemma 5.5. *Let* (H3) *hold and* $(V^\varepsilon,q^\varepsilon)$ *be the classical solution to the BSPDE* (5.13). *Then there exists a constant* $C>0$, *independent of* $\varepsilon>0$, *such that*

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^n}|V^\varepsilon_x(t,x)|^2dx+\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{|\sigma(r,x)^TV^\varepsilon_{xx}(r,x)+q^\varepsilon_x(r,x)^T|^2+\frac{|V^\varepsilon_x(r,x)|^2}{\varepsilon}\psi'\Big(\frac{V^\varepsilon(r,x)}{\varepsilon}\Big)\Big\}dxdr\\&\qquad\le C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big(|\bar g(r,x)|^2+|\bar g_x(r,x)|^2\Big)dxdr,\qquad\forall\varepsilon>0,t\in[0,T].\end{aligned}\tag{5.22}$$

Note that in [18] (see [19] also), to obtain an estimate similar to (5.19), a symmetric condition was assumed. Our proof above removes such a condition. Next result gives the monotonicity of the sequence $\{V^\varepsilon(\cdot\,,\cdot)\}_{\varepsilon>0}$ in $\varepsilon>0$.

Lemma 5.6. *Let* (H3) *hold and* $0<\varepsilon<\delta$. *Let* $(V^\varepsilon,q^\varepsilon)$ *be the adapted classical solution of BSPDE* (5.13), *and* (V^δ,q^δ) *be the adapted classical solution of* (5.13) *with* ε *replaced by* δ. *Then*

$$V^\varepsilon(t,x)\le V^\delta(t,x),\qquad(t,x)\in[0,T]\times\mathbb{R}^n,\ \text{a.s.}\tag{5.23}$$

Proof. We observe that by the definition of $\psi(\cdot)$, one has

$$\psi\Big(\frac{v}{\varepsilon}\Big)\ge\psi\Big(\frac{v}{\delta}\Big),\qquad\forall 0<\varepsilon<\delta,\quad v\in\mathbb{R}.\tag{5.24}$$

Hence, by letting

$$V^{\varepsilon,\delta}=V^\varepsilon-V^\delta,\qquad q^{\varepsilon,\delta}=q^\varepsilon-q^\delta,\tag{5.25}$$

we have

$$\begin{aligned}V^{\varepsilon,\delta}(t,x)=\int_t^T&\Big\{\frac{1}{2}\nabla\cdot\Big(\sigma\sigma^TV^{\varepsilon,\delta}_x\Big)+\langle\,\bar b,V^{\varepsilon,\delta}_x\,\rangle+\bar b^0V^{\varepsilon,\delta}+\nabla\cdot(\sigma q^{\varepsilon,\delta})+\langle\,\bar\sigma^0,q^{\varepsilon,\delta}\,\rangle\\&-\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)-\psi\Big(\frac{V^\delta}{\delta}\Big)\Big]\Big\}dr-\int_t^T\langle\,q^{\varepsilon,\delta},dW(r)\,\rangle,\qquad(t,x)\in[0,T]\times\mathbb{R}^n.\end{aligned}\tag{5.26}$$

Note that

$$\begin{aligned}-\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)-\psi\Big(\frac{V^\delta}{\delta}\Big)\Big]&=-\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)-\psi\Big(\frac{V^\delta}{\varepsilon}\Big)+\psi\Big(\frac{V^\delta}{\varepsilon}\Big)-\psi\Big(\frac{V^\delta}{\delta}\Big)\Big]\\&=-\Big[\int_0^1\psi'\Big(\frac{\lambda V^\varepsilon+(1-\lambda)V^\delta}{\varepsilon}\Big)d\lambda\Big]\frac{V^{\varepsilon,\delta}}{\varepsilon}-\Big[\psi\Big(\frac{V^\delta}{\varepsilon}\Big)-\psi\Big(\frac{V^\delta}{\delta}\Big)\Big],\end{aligned}\tag{5.27}$$

with

$$-\Big[\psi\Big(\frac{V^\delta}{\varepsilon}\Big)-\psi\Big(\frac{V^\delta}{\delta}\Big)\Big]\le 0,\tag{5.28}$$

Hence, by a comparison theorem for linear BSPDEs ([18]), we have

$$V^{\varepsilon,\delta}(t,x)\le 0,\qquad(t,x)\in[0,T]\times\mathbb{R}^n,\quad\text{a.s. },\tag{5.29}$$

which proves our conclusion. $\square$

Having estimates (5.18)–(5.20), (5.22) and (5.23) for $(V^\varepsilon, q^\varepsilon)$, we are now ready to prove the following result.

Theorem 5.7. *Let* (H3) *hold. Then BSPDVI* (4.20) *admits an adapted weak solution* $(\bar V, \bar q, \bar\zeta)$.

Proof. First of all, by Lemma 5.6, we see that

$$\lim_{\varepsilon\to 0} V^\varepsilon(t,x) = \bar V(t,x), \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^n, \text{ a.s.} \tag{5.30}$$

for some $\mathbb{F}$-adapted random field $\bar V(\cdot\,,\cdot)$ with $(t,x)\mapsto \bar V(t,x)$ being upper semicontinuous, almost surely. Next, by taking $2m$-th root in both sides of (5.18), and then sending $m\to\infty$, we get

$$\operatorname*{esssup}_{(t,x,\omega)\in[0,T]\times\mathbb{R}^n\times\Omega} |V^\varepsilon(t,x,\omega)| \le C \operatorname*{esssup}_{(t,x,\omega)\in[0,T]\times\mathbb{R}^n\times\Omega} |\bar g(t,x,\omega)|, \qquad \forall \varepsilon>0. \tag{5.31}$$

Hence, by taking $m=1$ in (5.18) and combining it with (5.22) and (5.31), we have

$$\begin{aligned}
&\operatorname*{esssup}_{(t,x,\omega)\in[0,T]\times\mathbb{R}^n\times\Omega} |V^\varepsilon(t,x,\omega)|^2 + \sup_{t\in[0,T]} \mathbb{E}\int_{\mathbb{R}^n} |V^\varepsilon_x(t,x)|^2 dx \\
&\quad + \mathbb{E}\int_0^T\!\!\int_{\mathbb{R}^n} \Big\{ |\sigma(r,x)^T V^\varepsilon_{xx}(r,x) + q^\varepsilon_x(r,x)^T|^2 + |q^\varepsilon(r,x)|^2 \\
&\qquad + \frac{V^\varepsilon(r,x)[V^\varepsilon(r,x)-\varepsilon] + |V^\varepsilon_x(r,x)|^2}{\varepsilon} I_{\{V^\varepsilon\ge 2\varepsilon\}} \Big\} dx dr \\
&\le C\Big[\operatorname*{esssup}_{(t,x,\omega)\in[0,T]\times\mathbb{R}^n\times\Omega} |\bar g(t,x,\omega)|^2 + \mathbb{E}\int_0^T\int_{\mathbb{R}^n} |\bar g_x(r,x)|^2 dx dr \Big], \qquad \forall\varepsilon>0.
\end{aligned} \tag{5.32}$$

Next, by (5.32) and (5.20), together with the above, we know that with the $\bar V$ as in (5.30), and for some $\bar q$ and $\bar\zeta$, one has

$$\begin{cases}
V^\varepsilon \to \bar V, & \text{strongly in } L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n)), \\
V^\varepsilon_x \to \bar V_x, & \text{weakly in } L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n;\mathbb{R}^{1\times n})), \\
q^\varepsilon \to \bar q, & \text{weakly in } L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n;\mathbb{R}^d)), \\
\psi\Big(\dfrac{V^\varepsilon}{\varepsilon}\Big) \to \bar\zeta, & \text{weakly in } L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n)),
\end{cases} \tag{5.33}$$

and

$$\mathbb{E}\int_0^T\int_{\mathbb{R}^n} \Big\{ [V^\varepsilon(r,x)-\varepsilon]^2 + |V^\varepsilon_x(r,x)|^2 \Big\} I_{(V^\varepsilon\ge 2\varepsilon)} dx dr \le C\varepsilon. \tag{5.34}$$

This yields

$$\begin{cases}
\Big[(V^\varepsilon-\varepsilon) + |V^\varepsilon_x|\Big] I_{\{V^\varepsilon\ge 2\varepsilon\}} \to 0, & \text{strongly in } L^2_{\mathbb{F}}(0,T;L^2(\mathbb{R}^n)), \\
\Big[V^\varepsilon(t,x) + |V^\varepsilon_x(t,x)|\Big] I_{\{V^\varepsilon\ge 2\varepsilon\}} \to 0, & \text{a.e. } (t,x)\in[0,T]\times\mathbb{R}^n, \text{ a.s.}
\end{cases} \tag{5.35}$$

Then it is necessary that

$$\bar V(t,x) \le 0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \quad \text{a.s. }, \tag{5.36}$$

and together with (5.30), we have

$$\{\bar V<0\} = \bigcup_{\varepsilon>0} \{V^\varepsilon<0\}. \tag{5.37}$$

Also, it is necessary that

$$\bar\zeta(t,x)\ge 0, \qquad \text{a.e. } (t,x)\in[0,T]\times\mathbb{R}^n, \text{ a.s.} \tag{5.38}$$

On the other hand, by applying the dominated convergence theorem to (5.19), we obtain

$$\begin{cases} (V^\varepsilon)^+\psi\Big(\dfrac{V^\varepsilon}{\varepsilon}\Big)=V^\varepsilon\psi\Big(\dfrac{V^\varepsilon}{\varepsilon}\Big)\to 0, & \text{strongly in } L^1_{\mathbb{F}}(0,T;L^1(\mathbb{R}^n)),\\ V^\varepsilon(t,x)\psi\Big(\dfrac{V^\varepsilon(t,x)}{\varepsilon}\Big)\to 0, & \text{a.e. } (t,x)\in[0,T]\times\mathbb{R}^n, \text{ a.s.}\end{cases}\tag{5.39}$$

Hence, it is necessary that

$$\bar V(t,x)\bar\zeta(t,x)=0,\qquad \text{a.e. } (t,x)\in[0,T]\times\mathbb{R}^n, \text{ a.s.}\tag{5.40}$$

Now, for any $\varphi(\cdot)\in C_0^\infty(\mathbb{R}^n)$, we have from (5.13) that

$$\begin{aligned}\int_{\mathbb{R}^n}V^\varepsilon(t,x)\varphi(x)dx=&\int_t^T\int_{\mathbb{R}^n}\Big\{-\langle\,\frac12\sigma^TV^\varepsilon_x+q^\varepsilon,\sigma^T\varphi_x\,\rangle+\Big[\,\langle\,\bar b,V^\varepsilon_x\,\rangle+\bar b^0V^\varepsilon\\ &+\langle\,\bar\sigma^0,q^\varepsilon\,\rangle+\bar g-\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big]\varphi\Big\}dxdr-\int_t^T\langle\int_{\mathbb{R}^n}q^\varepsilon\varphi dx,dW(r)\,\rangle,\quad t\in[0,T].\end{aligned}\tag{5.41}$$

Then letting $\varepsilon\to0$, along a sequence, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\bar V(t,x)\varphi(x)dx=&\int_t^T\int_{\mathbb{R}^n}\Big\{-\langle\,\frac12\sigma^T\bar V_x+\bar q,\sigma^T\varphi_x\,\rangle\\ &+\Big[\,\langle\,\bar b,\bar V_x\,\rangle+\bar b^0\bar V+\langle\,\bar\sigma^0,\bar q\,\rangle+\bar g-\bar\zeta\Big]\varphi\Big\}dxdr-\int_t^T\langle\int_{\mathbb{R}^n}\bar q\varphi dx,dW(r)\,\rangle,\quad t\in[0,T].\end{aligned}\tag{5.42}$$

Hence, $(\bar V,\bar q,\bar\zeta)$ is the adapted weak solution of (4.20). □

From the above, we see that for any $\varphi(\cdot)\in C_0^\infty(\mathbb{R}^n)$, the map $t\mapsto\int_{\mathbb{R}^n}\bar V(t,x)\varphi(x)dx$ is continuous.

6 Identification of the Value Function

In this section, we are going to identify the weak adapted solution of BSPDVI as the value function $V(\cdot\,,\cdot)$ of Problem (S). Suppose $(V^\varepsilon(\cdot),q^\varepsilon(\cdot))$ is the adapted classical solution of (5.13). Let

$$\begin{cases}\widetilde V^\varepsilon(t,x)=(1+|x|^p)V^\varepsilon(t,x)+h(t,x),\\ \widetilde q^{\,\varepsilon}(t,x)=(1+|x|^p)q^\varepsilon(t,x)+\mu(t,x),\end{cases}\qquad(t,x)\in[0,T]\times\mathbb{R}^n.\tag{6.1}$$

Then

$$\begin{aligned}\widetilde V^\varepsilon(t,x)=h(T,x)+&\int_t^T\Big\{\frac12\text{tr}\,\Big[\sigma(r,x)\sigma(r,x)^T\widetilde V^\varepsilon_{xx}(r,x)\Big]+\langle\,b(r,x),\widetilde V^\varepsilon_x(r,x)\,\rangle\\ &+\text{tr}\Big[\sigma(r,x)\widetilde q^{\,\varepsilon}_x(r,x)\Big]-\psi\Big(\frac{\widetilde V^\varepsilon(r,x)-h(r,x)}{\varepsilon}\Big)+g(r,x)\Big\}dr-\int_t^T\langle\,\widetilde q^{\,\varepsilon}(r,x),dW(r)\,\rangle\\ \equiv h(T,x)+&\int_t^T\widetilde q^{\,0,\varepsilon}(r,x)dr-\int_t^T\langle\,\widetilde q^{\,\varepsilon}(r,x),dW(r)\,\rangle,\qquad(t,x)\in[s,T]\times\mathbb{R}^n.\end{aligned}\tag{6.2}$$

Consequently, by Itô-Kunita's formula, we have

$$
\begin{aligned}
\widetilde V^\varepsilon(t,X(t;s,\xi)) = \widetilde V^\varepsilon(s,\xi) + & \int_s^t \Big\{ \widetilde q^{\,0,\varepsilon}(r,X(r;s,\xi)) + \langle\, b(r,X(r;s,\xi)), \widetilde V_x^\varepsilon(r,X(r;s,\xi))\,\rangle \\
& + \frac12 \mathrm{tr}\Big[\sigma(r,X(r;s,\xi))\sigma(r,X(r;s,\xi))^T \widetilde V_{xx}^\varepsilon(r,X(r;s,\xi))\Big] \\
& + \mathrm{tr}\Big[\sigma(r,X(r;s,\xi))\widetilde q_x^{\,\varepsilon}(r,X(r;s,\xi))\Big]\Big\} dr \\
& + \int_s^t \langle\, \widetilde q^{\,\varepsilon}(r,X(r;s,\xi)) + \sigma(r,X(r;s,\xi))^T \widetilde V_x^\varepsilon(r,X(r;s,\xi)), dW(r)\,\rangle \\
= \widetilde V^\varepsilon(s,\xi) + & \int_s^t \Big\{ \psi\Big(\frac{\widetilde V^\varepsilon(r,X(r;s,\xi)) - h(r,X(r;s,\xi))}{\varepsilon}\Big) - g(r,X(r;s,\xi))\Big\} dr \\
& + \int_s^t \langle\, \widetilde q^{\,\varepsilon}(r,X(r;s,\xi)) + \sigma(r,X(r;s,\xi))^T \widetilde V_x^\varepsilon(r,X(r;s,\xi)), dW(r)\,\rangle\,.
\end{aligned}
\tag{6.3}
$$

Thus, for any $\tau \in \mathcal S[s,T]$,

$$
\begin{aligned}
J_{s,\xi}(\tau) &= \mathbb E\Big[\int_s^\tau g(r,X(r;s,\xi))dr + h(\tau,X(\tau;s,\xi))\big|\mathcal F_s\Big] \\
&= \widetilde V^\varepsilon(s,\xi) + \mathbb E\Big[\int_s^\tau \psi\Big(\frac{\widetilde V^\varepsilon(r,X(r;s,\xi)) - h(r,X(r;s,\xi))}{\varepsilon}\Big) dr\big|\mathcal F_s\Big] \\
&\quad + \mathbb E\Big[h(\tau,X(\tau;s,\xi)) - \widetilde V^\varepsilon(\tau,X(\tau,s,\xi))\big|\mathcal F_s\Big].
\end{aligned}
\tag{6.4}
$$

By our discussion above, we know that under (H3),

$$
\widetilde V^\varepsilon(t,x) \downarrow V^*(t,x), \qquad \forall (t,x)\in[0,T]\times\mathbb R^n, \ \text{a.s.}\ , \tag{6.5}
$$

with $(t,x)\mapsto V^*(t,x)$ being upper semi-continuous, and by (5.36),

$$
V^*(s,\xi) = (1+|x|^p)\bar V(s,\xi) + h(s,\xi) \le h(s,\xi), \qquad \text{a.s.}\ , \ \forall (s,\xi)\in\mathcal D^p. \tag{6.6}
$$

Now, sending $\varepsilon\to 0$ in (6.4), we have

$$
J_{s,\xi}(\tau) = V^*(s,\xi) + \mathbb E\Big[\int_s^\tau \zeta^*(r,X(r;s,\xi))dr\big|\mathcal F_s\Big] + \mathbb E\Big[h(\tau,X(\tau;s,\xi)) - V^*(\tau,X(\tau,s,\xi))\big|\mathcal F_s\Big], \tag{6.7}
$$

with

$$
\zeta^*(t,x)\in\beta\Big(V^*(t,x)-h(t,x)\Big), \qquad (t,x)\in[0,T]\times\mathbb R^n, \ \text{a.s.} \tag{6.8}
$$

Hence,

$$
J_{s,\xi}(\tau) \ge V^*(s,\xi), \qquad (s,\xi)\in\mathcal D^p, \quad \tau\in\mathcal S[s,T], \tag{6.9}
$$

which leads to

$$
V(s,\xi) \ge V^*(s,\xi), \qquad (s,\xi)\in\mathcal D^p. \tag{6.10}
$$

Next, let $(s,\xi)\in\mathcal D^p$ be fixed. We define

$$
\tau^*(s,\xi) = \inf\Big\{ r\in[s,T] \bigm| V^*(r,X(r;s,\xi)) = h(r,X(r;s,\xi))\Big\}. \tag{6.11}
$$

Since $t\mapsto V^*(r,X(r;s,\xi)) - h(r,X(r;s,\xi))$ is $\mathbb F$-progressively measurable, by Début Theorem (see [6], [21]), $\tau^*(s,\xi)\in\mathcal S[s,T]$. Then taking $\tau=\tau^*(s,\xi)$ in (6.7), we obtain

$$
V(s,\xi) \le J_{s,\xi}(\tau^*(s,\xi)) = V^*(s,\xi). \tag{6.12}
$$

Hence, combining the above with (6.10), one must have the following:

$$V^*(s,\xi) = V(s,\xi), \qquad \forall (s,\xi) \in \mathcal{D}^p. \tag{6.13}$$

That means $V^*(\cdot\,,\cdot)$ must be the value function $V(\cdot\,,\cdot)$ of Problem (S). Consequently, $(t,x) \mapsto V^*(t,x) = V(t,x)$ must be continuous itself. Moreover, the smallest optimal stopping time can be identified through (3.4).

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7 Appendix

In this appendix, we collect some proofs.

Sketch proof of Theorem 3.1. (i) For any $\tau \in \mathcal{S}[s,T]$, we have

$$V(s,\xi) \le J_{s,\xi}(\tau) \equiv \mathbb{E}\Big[\int_s^\tau g(r,X(r;s,\xi))dr + h(\tau,X(\tau))\big|\mathcal{F}_s\Big]. \tag{7.1}$$

In particular, taking $\tau = s$, one obtains (3.1). Next, for any $\tau \in \mathcal{S}[s,T]$, take $\theta \in \mathcal{S}[\tau,T]$. One has

$$V(s,\xi) \le J_{s,\xi}(\theta) = \mathbb{E}\Big[\int_s^\tau g(r,X(r;s,\xi))dt + J_{\tau,X(\tau;s,\xi)}(\theta)\big|\mathcal{F}_s\Big]. \tag{7.2}$$

Taking infimum with respect to $\theta \in \mathcal{S}[\tau,T]$ yields

$$V(s,\xi) \le \mathbb{E}\Big[\int_s^\tau g(r,X(r;s,\xi))dr + V(\tau,X(\tau;s,\xi))\big|\mathcal{F}_s\Big]. \tag{7.3}$$

Hence, (3.2) holds.

(ii) Suppose $\bar\theta \in \mathcal{S}[\theta,T]$ is optimal for the initial point $(s,\xi)\in\mathcal{D}^p$. Then

$$\begin{aligned} V(s,\xi) = J_{s,\xi}(\bar\theta) &\geq \mathbb{E}\Big[\int_s^{\bar\theta} g(r,X(r;s,\xi))dr + V(\bar\theta, X(\bar\theta;s,\xi))\big|\mathcal{F}_s\Big] \\ &\geq \inf_{\tau\in\mathcal{S}[s,T]} \mathbb{E}\Big[\int_s^{\tau} g(r,X(r;s,\xi))dr + V(\tau, X(\tau;s,\xi))\big|\mathcal{F}_s\Big] \geq V(s,\xi). \end{aligned} \tag{7.4}$$

Hence, the equalities in the above have to hold, which implies

$$\mathbb{E}\Big[V(\bar\theta, X(\bar\theta;s,\xi))\big|\mathcal{F}_s\Big] = \mathbb{E}\Big[h(\bar\theta, X(\bar\theta;s,\xi))\big|\mathcal{F}_s\Big]. \tag{7.5}$$

Combining the fact

$$V(\bar\theta, X(\bar\theta;s,\xi)) \leq h(\bar\theta, X(\bar\theta;s,\xi)), \qquad \text{a.s. },$$

we obtain (3.3). Next, for (3.5), if there exists a $\Omega_0 \subseteq \{V(s,\xi) < h(s,\xi)\}$ with $\mathbb{P}(\Omega_0) > 0$ such that

$$\bar\tau(s,\xi) = s, \qquad \text{on } \ \Omega_0. \tag{7.6}$$

then, trivially,

$$V(s,\xi(\omega)) = h(s,\xi(\omega)), \qquad \omega\in\Omega_0, \tag{7.7}$$

which contradicts the choice of Ω_0. Conversely, if $\Omega_0 \subseteq \{\bar\tau(s,\xi) > s\}$ with $\mathbb{P}(\Omega_0) > 0$ such that (7.7) holds, then (7.6) has to be true (by definition of $\bar\tau(s,\xi)$), a contradiction to the choice of Ω_0. Hence, (3.5) holds.

We now show (3.6). To this end, let $(s,\xi)\in\mathcal{D}^p$. Define $\bar\tau(s,\xi)$ by (3.4) and suppose $\mathbb{P}\{s<\bar\tau(s,\xi)\}>0$. The case $\theta=\bar\tau(s,\xi)$ is trivial. Thus, we fix a $\theta\in\mathcal{S}[s,\bar\tau(s,\xi))$, and let $\tau\in\mathcal{S}[\theta,\bar\tau(s,\xi)]$. From (3.3), we know that any $\mu\in\mathcal{S}[\theta,T]$ with $\mathbb{P}\{\mu<\tau\}>0$ is not optimal for the initial point $(\theta, X(\theta;s,\xi))$. Hence,

$$\begin{aligned} V(\theta,X(\theta;s,\xi)) &= \inf_{\mu\in\mathcal{S}[\tau,T]} \mathbb{E}\Big[\int_\theta^{\tau} g(r,X(r;s,\xi))dr + J_{\tau,X(\tau;s,\xi)}(\mu)\big|\mathcal{F}_\theta\Big] \\ &= \mathbb{E}\Big[\int_\theta^{\tau} g(r,X(r;s,\xi))dr + V(\tau,X(\tau;s,\xi))\big|\mathcal{F}_\theta\Big], \end{aligned} \tag{7.8}$$

proving (3.6).

Finally, by taking $\theta=s$ and $\tau=\bar\tau(s,\xi)$ in (3.6), we see that

$$V(s,\xi) = \mathbb{E}\Big[\int_s^{\bar\tau(s,x)} g(s,X(s;s,\xi))ds + h(\bar\tau(s,\xi), X(\bar\tau(s,x);s,\xi))\big|\mathcal{F}_s\Big] = J_{s,\xi}(\bar\tau(s,\xi)), \tag{7.9}$$

which means that $\bar\tau(s,\xi)$ is an optimal stopping time of Problem (S) for the initial point (s,ξ), and it must be the smallest one. $\square$

Proof of Lemma 5.3. By (5.16), we may assume that

$$\frac{\theta'(\rho)}{\theta''(\rho)} = \frac{\theta'(\rho)}{\theta''(\rho)} I_{\{\theta''(\rho)>0\}}, \qquad \forall \rho\in\mathbb{R}, \tag{7.10}$$

since $\theta : \mathbb{R} \to [0, \infty)$ is convex and piecewise smooth. Applying Itô's formula to $\theta(V^\varepsilon)$, we have

$$
\begin{aligned}
&-\mathbb{E}\int_{\mathbb{R}^n}\theta\Big(V^\varepsilon(t,x)\Big)dx = -\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\theta'(V^\varepsilon)\Big[\frac{1}{2}\nabla\cdot\Big(\sigma\sigma^T V_x^\varepsilon\Big) + \langle\,\bar b, V_x^\varepsilon\,\rangle \\
&\qquad +\bar b^0 V^\varepsilon + \nabla\cdot(\sigma q^\varepsilon) + \langle\,\bar\sigma^0, q^\varepsilon\,\rangle + \bar g - \psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big] - \frac{1}{2}\theta''(V^\varepsilon)|q^\varepsilon|^2\Big\}dxdr \\
&= \frac{1}{2}\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\theta''(V^\varepsilon)|\sigma^T V_x^\varepsilon|^2 + 2\,\langle\, q^\varepsilon, \theta''(V^\varepsilon)\sigma^T V_x^\varepsilon - \theta'(V^\varepsilon)\bar\sigma^0\,\rangle + \theta''(V^\varepsilon)|q^\varepsilon|^2 \\
&\qquad +2(\nabla\cdot\bar b)\theta(V^\varepsilon) + 2\theta'(V^\varepsilon)\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) - \bar b^0 V^\varepsilon - \bar g\Big]\Big\}dxdr \\
&= \frac{1}{2}\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\theta''(V^\varepsilon)\Big[|\sigma^T V_x^\varepsilon|^2 + 2\,\langle\, q^\varepsilon, \sigma^T V_x^\varepsilon - \frac{\theta'(V^\varepsilon)}{\theta''(V^\varepsilon)}\bar\sigma^0\,\rangle + |q^\varepsilon|^2\Big] \\
&\qquad +2(\nabla\cdot\bar b)\theta(V^\varepsilon) + 2\theta'(V^\varepsilon)\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) - \bar b^0 V^\varepsilon - \bar g\Big]\Big\}dxdr \qquad (7.11)\\
&= \frac{1}{2}\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\theta''(V^\varepsilon)\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon - \frac{\theta'(V^\varepsilon)}{\theta''(V^\varepsilon)}\bar\sigma^0\Big|^2 + 2\Big[(\nabla\cdot\bar b) - \nabla\cdot(\sigma\bar\sigma^0)\Big]\theta(V^\varepsilon) \\
&\qquad -\frac{|\theta'(V^\varepsilon)|^2}{\theta''(V^\varepsilon)}|\bar\sigma^0|^2 + \theta'(V^\varepsilon)\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) - \bar b^0 V^\varepsilon - \bar g\Big]\Big\}dxdr \\
&\geq \frac{1}{2}\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\frac{\theta''(V^\varepsilon)}{2}\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon\Big|^2 + 2\Big[(\nabla\cdot\bar b) - \nabla\cdot(\sigma\bar\sigma^0)\Big]\theta(V^\varepsilon) \\
&\qquad -\frac{2|\theta'(V^\varepsilon)|^2}{\theta''(V^\varepsilon)}|\bar\sigma^0|^2 + \theta'(V^\varepsilon)\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) - \bar b^0 V^\varepsilon - \bar g\Big]\Big\}dxdr \\
&\geq \frac{1}{2}\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\theta''(V^\varepsilon)\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon\Big|^2 + \theta'(V^\varepsilon)\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) - C\theta(V^\varepsilon) - C|\theta'(V^\varepsilon)|\Big(|V^\varepsilon| + |\bar g|\Big)\Big\}dxdr.
\end{aligned}
$$

In the above, we have used the fact that $|a-b| \geq \frac{1}{2}|a| - |b|$. Note that under our conditions,

$$
\theta'(\rho)\psi\Big(\frac{\rho}{\varepsilon}\Big) \geq 0, \qquad \text{a.e. } \rho \in \mathbb{R}. \tag{7.12}
$$

Hence, by Gronwall's inequality, we obtain (5.17). $\square$

Proof of Lemma 5.4. For any $m \geq 1$, taking

$$
\theta(\rho) = |\rho|^{2m}, \qquad \rho \in \mathbb{R}^n. \tag{7.13}
$$

Then (5.15) and (5.16) hold. Hence, by Lemma 5.3, we obtain

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n} V^\varepsilon(t,x)^{2m}dx + \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{2m(2m-1)(V^\varepsilon)^{2m-2}\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon\Big|^2 \\
&\qquad +2m(V^\varepsilon)^{2m-1}\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big\}dxdr \leq C\mathbb{E}\int_t^T\int_{\mathbb{R}^n} 2m|V^\varepsilon|^{2m-1}\Big(|V^\varepsilon| + |\bar g|\Big)\Big\}dxdr \qquad (7.14)\\
&\qquad \leq C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{(4m-1)(V^\varepsilon)^{2m} + |\bar g|^{2m}\Big\}dxdr.
\end{aligned}
$$

Then by Gronwall's inequality, one has

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n} V^\varepsilon(t,x)^{2m}dx + \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{2m(2m-1)(V^\varepsilon)^{2m-2}\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon\Big|^2 \\
&\qquad +2m(V^\varepsilon)^{2m-1}\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big\}dxdr \leq Ce^{Cm}\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\bar g^{2m}dxdr,
\end{aligned} \tag{7.15}
$$

which implies (5.18).

Next, by taking

$$\theta(\rho) = (\rho^+)^2, \qquad \rho \in \mathbb{R}, \tag{7.16}$$

we have

$$\theta'(\rho) = 2\rho^+, \quad \theta''(\rho) = 2I_{\{\rho>0\}}, \tag{7.17}$$

which leads to

$$\theta'(\rho)^2 = 4(\rho^+)^2 = 2\theta(\rho)\theta''(\rho). \tag{7.18}$$

Thus, (5.15) and (5.16) hold. Hence, by Lemma 5.3, we have

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n}\Big(V^\varepsilon(t,x)^+\Big)^2 dx + \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{2I_{\{V^\varepsilon>0\}}\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon\Big|^2 + 2(V^\varepsilon)^+\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big\}dxdr \\
&\le C\mathbb{E}\int_t^T\int_{\mathbb{R}^n} 2(V^\varepsilon)^+|\Big[|V^\varepsilon| + |\bar g|\Big]\Big\}dxdr \le C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big[\Big((V^\varepsilon)^+\Big)^2 + |\bar g|^2 I_{\{V^\varepsilon>0\}}\Big]\Big\}dxdr.
\end{aligned} \tag{7.19}$$

It follows from Gronwall's inequality that

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^n}\Big(V^\varepsilon(t,x)^+\Big)^2 dx + \mathbb{E}\int_0^T\int_{\mathbb{R}^n}\Big\{\Big|\,\sigma^T V_x^\varepsilon + q^\varepsilon\Big|^2 I_{\{V^\varepsilon>0\}} + (V^\varepsilon)^+\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big\}dxdr \\
\le C\mathbb{E}\int_0^T\int_{\mathbb{R}^n}|\bar g(r,x)|^2 I_{\{V^\varepsilon>0\}}dxdr, \qquad \forall \varepsilon>0,
\end{aligned} \tag{7.20}$$

with $C>0$ independent of $\varepsilon>0$, which leads to (5.19).

Finally, we take

$$\theta(\rho) = \rho\psi\Big(\frac{\rho}{\varepsilon}\Big) = \frac{\rho^2}{\varepsilon}\int_0^{\rho/\varepsilon}\eta(r)dr - \rho\int_0^{\rho/\varepsilon} r\eta(r)dr.$$

Then

$$\begin{aligned}
0 \le \theta'(\rho) &= \psi\Big(\frac{\rho}{\varepsilon}\Big) + \frac{\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) = \frac{\rho}{\varepsilon}\int_0^{\rho/\varepsilon}\eta(r)dr - \int_0^{\rho/\varepsilon} r\eta(r)dr + \frac{\rho}{\varepsilon}\int_0^{\rho/\varepsilon}\eta(r)dr \\
&= \frac{2\rho}{\varepsilon}\int_0^{\rho/\varepsilon}\eta(r)dr - \int_0^{\rho/\varepsilon} r\eta(r)dr,
\end{aligned} \tag{7.21}$$

and

$$0 \le \theta''(\rho) = \frac{2}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) + \frac{\rho}{\varepsilon^2}\psi''\Big(\frac{\rho}{\varepsilon}\Big) = \frac{2}{\varepsilon}\int_0^{\rho/\varepsilon}\eta(r)dr + \frac{\rho}{\varepsilon^2}\eta\Big(\frac{\rho}{\varepsilon}\Big). \tag{7.22}$$

Note that

$$\theta'(\rho)^2 - C\theta(\rho)\theta''(\rho) = \Big[\psi\Big(\frac{\rho}{\varepsilon}\Big) + \frac{\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big)\Big]^2 - C\psi\Big(\frac{\rho}{\varepsilon}\Big)\Big[\frac{2\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) + \frac{\rho^2}{\varepsilon^2}\psi''\Big(\frac{\rho}{\varepsilon}\Big)\Big]. \tag{7.23}$$

Now, for $\rho \in (-\infty, 0]$, we have

$$\theta'(\rho)^2 - C\theta(\rho)\theta''(\rho) = 0, \tag{7.24}$$

for any $C>0$; For $\rho \in (0,\varepsilon]$,

$$\theta'(\rho)^2 - C\theta(\rho)\theta''(\rho) = \Big[\frac{\rho^3}{6\varepsilon^3} + \frac{\rho}{\varepsilon}\frac{\rho^2}{2\varepsilon^2}\Big]^2 - C\frac{\rho^3}{6\varepsilon^3}\Big[\frac{2\rho}{\varepsilon}\frac{\rho^2}{2\varepsilon^2} + \frac{\rho^3}{\varepsilon^3}\Big] = \frac{\rho^6}{9\varepsilon^6}(4-3C) \le 0, \tag{7.25}$$

provided $C \ge \frac{4}{3}$; For $\rho \in (\varepsilon, 2\varepsilon]$, since both $\psi(\cdot)$ and $\psi'(\cdot)$ are nondecreasing,

$$\begin{aligned}
\theta'(\rho)^2 - C\theta(\rho)\theta''(\rho) &= \Big[\psi\Big(\frac{\rho}{\varepsilon}\Big) + \frac{\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big)\Big]^2 - C\psi\Big(\frac{\rho}{\varepsilon}\Big)\Big[\frac{2\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) + \frac{\rho^2}{\varepsilon^2}\psi''\Big(\frac{\rho}{\varepsilon}\Big)\Big] \\
&\le \Big[\psi(2) + 2\psi'(2)\Big]^2 - 2C\psi(1)\psi'(1) = 9 - \frac{C}{6} \le 0,
\end{aligned} \tag{7.26}$$

provided $C \geq 54$; And for $\rho \in [2\varepsilon, \infty)$,

$$\begin{aligned}&\theta'(\rho)^2 - C\theta(\rho)\theta''(\rho) = \Big[\frac{2\rho}{\varepsilon} - 1\Big]^2 - C\Big(\frac{\rho}{\varepsilon} - 1\Big)\frac{2\rho}{\varepsilon} \leq \frac{4\rho^2}{\varepsilon^2} - \frac{4\rho}{\varepsilon} + 1 - 2C\frac{\rho^2}{\varepsilon^2} + 2C\frac{\rho}{\varepsilon}\\&= -(C-2)\frac{2\rho}{\varepsilon}\Big(\frac{\rho}{\varepsilon} - 1\Big) + 1 \leq -4(C-2) + 1 = -4C + 9 \leq 0,\end{aligned} \tag{7.27}$$

provided $C \geq \frac{9}{4}$. Further, we claim that

$$\frac{\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) \leq 3\psi\Big(\frac{\rho}{\varepsilon}\Big) + 1, \qquad \rho \in \mathbb{R}. \tag{7.28}$$

In fact, the above holds for $\rho \leq 0$. Now, for $\rho \in (0, \varepsilon]$,

$$\frac{\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) = \frac{\rho^3}{2\varepsilon^3} = 3\psi\Big(\frac{\rho}{\varepsilon}\Big), \tag{7.29}$$

and for $\rho \in (\varepsilon, \infty)$,

$$\frac{\rho}{\varepsilon}\psi'\Big(\frac{\rho}{\varepsilon}\Big) = \frac{\rho}{\varepsilon}\Big[1 - \frac{(2-\rho)^2}{2}I_{\{\rho \leq 2\varepsilon\}}\Big] \leq \frac{\rho}{\varepsilon} = \psi\Big(\frac{\rho}{\varepsilon}\Big) + 1. \tag{7.30}$$

Hence, by Lemma 5.3,

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^n} V^\varepsilon(t,x)\psi\Big(\frac{V^\varepsilon(t,x)}{\varepsilon}\Big)dx + \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{\Big[\frac{2}{\varepsilon}\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big) + \frac{V^\varepsilon}{\varepsilon^2}\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big]\Big|\sigma^T V^\varepsilon_x + q^\varepsilon\Big|^2\\&\qquad\qquad + \Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) + \frac{V^\varepsilon}{\varepsilon}\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big]\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big\}dxdr\\&\leq C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big[\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big) + \frac{V^\varepsilon}{\varepsilon}\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big]\Big(|V^\varepsilon| + |\bar g|\Big)dxdr\\&\leq C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\Big(V^\varepsilon + |\bar g|\Big)dxdr, \qquad t \in [0,T].\end{aligned} \tag{7.31}$$

Then by Gronwall's inequality, together with Cauchy-Schwartz inequality, we obtain

$$\mathbb{E}\int_{\mathbb{R}^n} V^\varepsilon(t,x)\psi\Big(\frac{V^\varepsilon(t,x)}{\varepsilon}\Big)dx + \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\psi\Big(\frac{V^\varepsilon}{\varepsilon}\Big)^2 dxdr \leq C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}|\bar g|^2 dxdr, \quad t \in [0,T]. \tag{7.32}$$

This leads to (5.20). □

Proof of Lemma 5.5. Applying Itô's formula to $|V^\varepsilon_{x_k}(t,x)|^2$ yields

$$\begin{aligned}&-\mathbb{E}\int_{\mathbb{R}^n}|V^\varepsilon_{x_k}(t,x)|^2 dx\\&= -\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{2V^\varepsilon_{x_k}\Big[\frac{1}{2}\nabla\cdot\Big(\sigma\sigma^T(V^\varepsilon_{x_k})_x\Big) + \langle\, \bar b, (V^\varepsilon_{x_k})_x\,\rangle + \bar b^0 V^\varepsilon_{x_k} + \nabla\cdot(\sigma q^\varepsilon_{x_k}) + \langle\, \bar\sigma^0, q^\varepsilon_{x_k}\,\rangle\\&\qquad -\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{V^\varepsilon_{x_k}}{\varepsilon} + \frac{1}{2}\nabla\cdot\Big((\sigma\sigma^T)_{x_k}V^\varepsilon_x\Big) + \langle\, \bar b_{x_k}, V^\varepsilon_x\,\rangle + \bar b^0_{x_k}V^\varepsilon\\&\qquad + \nabla\cdot(\sigma_{x_k}q^\varepsilon) + \langle\, \bar\sigma^0_{x_k}, q^\varepsilon\,\rangle + \bar g_{x_k}\Big] - |q^\varepsilon_{x_k}|^2\Big\}dxdr\\&= \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{|\sigma^T(V^\varepsilon_{x_k})_x|^2 + 2\,\langle\, q^\varepsilon_{x_k}, \sigma^T(V^\varepsilon_{x_k})_x - V^\varepsilon_{x_k}\bar\sigma^0\,\rangle + |q^\varepsilon_{x_k}|^2 + 2\,\langle\, q^\varepsilon, \sigma^T_{x_k}(V^\varepsilon_{x_k})_x - V^\varepsilon_{x_k}\bar\sigma^0_{x_k}\,\rangle\\&\qquad + \Big[(\nabla\cdot\bar b) - 2\bar b^0\Big](V^\varepsilon_{x_k})^2 - 2V^\varepsilon_{x_k}\bar g_{x_k} + 2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{(V^\varepsilon_{x_k})^2}{\varepsilon} + \langle (V^\varepsilon_x)_{x_k}, (\sigma\sigma^T)_{x_k}V^\varepsilon_x\,\rangle\\&\qquad - 2V^\varepsilon_{x_k}\,\langle\, \bar b_{x_k}, V^\varepsilon_x\,\rangle + \bar b^0_{x_k x_k}(V^\varepsilon)^2\Big\}dxdr.\end{aligned} \tag{7.33}$$

Note that (recalling $\sigma = (\sigma_1, \cdots, \sigma_d)$, with each σ_i taking values in $\mathbb{R}^n$)

$$
\begin{aligned}
&\nabla \cdot \Big[V^\varepsilon_{x_k}\sigma_{x_k}\Big] = \Big(\nabla \cdot [V^\varepsilon_{x_k}(\sigma_1)_{x_k}], \cdots, \nabla \cdot [V^\varepsilon_{x_k}(\sigma_d)_{x_k}]\Big)^T \\
&= \Big(\langle (V^\varepsilon_{x_k})_x, (\sigma_1)_{x_k}\rangle + V^\varepsilon_{x_k}\nabla \cdot [(\sigma_1)_{x_k}], \cdots, \langle (V^\varepsilon_{x_k})_x, (\sigma_d)_{x_k}\rangle + V^\varepsilon_{x_k}\nabla \cdot [(\sigma_d)_{x_k}]\Big)^T \\
&= \sigma^T_{x_k}(V^\varepsilon_{x_k})_x + V^\varepsilon_{x_k}\nabla \cdot (\sigma_{x_k}).
\end{aligned} \tag{7.34}
$$

Hence, (recall that q^ε_x takes values in $\mathbb{R}^{d\times n}$)

$$
\begin{aligned}
&\mathbb{E}\int_t^T \int_{\mathbb{R}^n} \langle q^\varepsilon, \sigma^T_{x_k}(V^\varepsilon_{x_k})_x\rangle\, dxdr = \mathbb{E}\int_t^T \int_{\mathbb{R}^n} \langle q^\varepsilon, \nabla \cdot \Big[V^\varepsilon_{x_k}\sigma_{x_k}\Big] - V^\varepsilon_{x_k}\nabla \cdot \sigma_{x_k}\rangle\, dxdr \\
&= -\mathbb{E}\int_t^T \int_{\mathbb{R}^n} V^\varepsilon_{x_k}\Big\{\mathrm{tr}\Big(\sigma_{x_k}q^\varepsilon_x\Big) + \langle q^\varepsilon, \nabla \cdot \sigma_{x_k}\rangle\Big\}dxdr.
\end{aligned} \tag{7.35}
$$

On the other hand,

$$
\begin{aligned}
&\Big[\langle (\sigma\sigma^T)_{x_k}V^\varepsilon_x, V^\varepsilon_x\rangle\Big]_{x_k} = \langle (\sigma\sigma^T)_{x_kx_k}V^\varepsilon_x, V^\varepsilon_x\rangle + \langle (\sigma\sigma^T)_{x_k}(V^\varepsilon_x)_{x_k}, V^\varepsilon_x\rangle + \langle (\sigma\sigma^T)_{x_k}V^\varepsilon_x, (V^\varepsilon_x)_{x_k}\rangle \\
&= \langle (\sigma\sigma^T)_{x_kx_k}V^\varepsilon_x, V^\varepsilon_x\rangle + 2\langle (\sigma\sigma^T)_{x_k}V^\varepsilon_x, (V^\varepsilon_x)_{x_k}\rangle,
\end{aligned} \tag{7.36}
$$

which implies that

$$
\mathbb{E}\int_t^T \int_{\mathbb{R}^n} \langle (V^\varepsilon_x)_{x_k}, (\sigma\sigma^T)_{x_k}V^\varepsilon_x\rangle\, dxdr = -\frac{1}{2}\mathbb{E}\int_t^T \int_{\mathbb{R}^n} \langle (\sigma\sigma^T)_{x_kx_k}V^\varepsilon_x, V^\varepsilon_x\rangle\, dxdr. \tag{7.37}
$$

Thus, (7.33) can be written as

$$
\begin{aligned}
&-\mathbb{E}\int_{\mathbb{R}^n} |V^\varepsilon_{x_k}(t,x)|^2 dx \\
&= \mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{|\sigma^T(V^\varepsilon_{x_k})_x|^2 + 2\langle q^\varepsilon_{x_k}, \sigma^T(V^\varepsilon_{x_k})_x - V^\varepsilon_{x_k}\bar\sigma^0\rangle + |q^\varepsilon_{x_k}|^2 - 2V^\varepsilon_{x_k}\mathrm{tr}\Big(\sigma_{x_k}q^\varepsilon_x\Big) \\
&\qquad + 2\langle q^\varepsilon, \bar\sigma^0_{x_k} - \nabla \cdot \sigma_{x_k}\rangle V^\varepsilon_{x_k} + [(\nabla \cdot \bar b) - 2\bar b^0](V^\varepsilon_{x_k})^2 - 2V^\varepsilon_{x_k}\bar g_{x_k} + 2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{(V^\varepsilon_{x_k})^2}{\varepsilon} \\
&\qquad - \frac{1}{2}\langle (\sigma\sigma^T)_{x_kx_k}V^\varepsilon_x, V^\varepsilon_x\rangle - 2V^\varepsilon_{x_k}\langle \bar b_{x_k}, V^\varepsilon_x\rangle + \bar b^0_{x_kx_k}(V^\varepsilon)^2\Big\}dxdr.
\end{aligned} \tag{7.38}
$$

In what follows, we let

$$
\langle A, B\rangle = \mathrm{tr}\,(AB^T) = \sum_{k=1}^n a_k^T b_k, \qquad \forall A \equiv \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix},\ B \equiv \begin{pmatrix} b_1^T \\ \vdots \\ b_n^T \end{pmatrix} \in \mathbb{R}^{n\times m}.
$$

Then one has

$$
|A|^2 = \mathrm{tr}\,(A^TA) = \sum_{i=1}^n\sum_{j=1}^m |a_{ij}|^2, \qquad \forall A \equiv (a_{ij}) \in \mathbb{R}^{n\times m}.
$$

Now, summing (7.38) up with respect to k, we obtain (recall that q_x takes values in $\mathbb{R}^{d\times n}$)

$$
\begin{aligned}
&-\mathbb{E}\int_{\mathbb{R}^n}|V_x^\varepsilon(t,x)|^2dx\\
&=\sum_{k=1}^n\mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^n}\Big\{|\sigma^T(V_{x_k}^\varepsilon)_x|^2+2\,\langle\, q_{x_k}^\varepsilon,\sigma^T(V_{x_k}^\varepsilon)_x-V_{x_k}^\varepsilon\bar\sigma^0\,\rangle+|q_{x_k}^\varepsilon|^2-2V_{x_k}^\varepsilon\mathrm{tr}\Big(\sigma_{x_k}q_x^\varepsilon\Big)\\
&\qquad+2\,\langle\, q^\varepsilon,\bar\sigma_{x_k}^0-\nabla\cdot\sigma_{x_k}\,\rangle\, V_{x_k}^\varepsilon+[(\nabla\cdot\bar b)-2\bar b^0](V_{x_k}^\varepsilon)^2-2V_{x_k}^\varepsilon\bar g_{x_k}+2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{(V_{x_k}^\varepsilon)^2}{\varepsilon}\\
&\qquad-\frac12\,\langle(\sigma\sigma^T)_{x_kx_k}V_x^\varepsilon,V_x^\varepsilon\,\rangle-2V_{x_k}^\varepsilon\,\langle\,\bar b_{x_k},V_x^\varepsilon\,\rangle+\bar b^0_{x_kx_k}(V^\varepsilon)^2\Big\}dxdr\\
&=\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{|\sigma^TV_{xx}^\varepsilon|^2+2\,\mathrm{tr}\Big[(q_x^\varepsilon)^T\sigma^TV_{xx}^\varepsilon\Big]+|q_x^\varepsilon|^2-2\,\mathrm{tr}\Big[(q_x^\varepsilon)^T\Big(\bar\sigma^0V_x^\varepsilon-\sum_{k=1}^nV_{x_k}^\varepsilon\sigma_{x_k}^T\Big)\Big]\\
&\qquad+2\,\langle\, q^\varepsilon,\Big(\bar\sigma^0-\nabla\cdot\sigma\Big)_xV_x^\varepsilon\,\rangle+\Big[(\nabla\cdot\bar b)-2\bar b^0\Big]|V_x^\varepsilon|^2-2\,\langle\, V_x^\varepsilon,\bar g_x\,\rangle+2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{|V_x^\varepsilon|^2}{\varepsilon}\\
&\qquad-\frac12\,\langle[\Delta(\sigma\sigma^T)]V_x^\varepsilon,V_x^\varepsilon\,\rangle-2\,\langle\,\bar b_xV_x^\varepsilon,V_x^\varepsilon\,\rangle+(\Delta\bar b^0)|V^\varepsilon|^2\Big\}dxdr\\
&=\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big\{|\sigma^TV_{xx}^\varepsilon+q_x^\varepsilon-\bar\sigma^0V_x^\varepsilon-\sum_{k=1}^nV_{x_k}^\varepsilon\sigma_{x_k}^T|^2-|\bar\sigma^0V_x^\varepsilon+\sum_{k=1}^nV_{x_k}^\varepsilon\sigma_{x_k}^T|^2\\
&\qquad+2\,\mathrm{tr}\Big[V_{xx}^\varepsilon\Big(\sigma\bar\sigma^0V_x^\varepsilon+\sum_{k=1}^nV_{x_k}^\varepsilon\sigma\sigma_{x_k}^T\Big)\Big]+2\,\langle\, q^\varepsilon,\Big(\bar\sigma^0-\nabla\cdot\sigma\Big)_xV_x^\varepsilon\,\rangle-2\,\langle\, V_x^\varepsilon,\bar g_x\,\rangle\\
&\qquad+2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{|V_x^\varepsilon|^2}{\varepsilon}-\langle\,\Big[\frac12\Delta(\sigma\sigma^T)]+\bar b_x+\bar b_x^T-\Big((\nabla\cdot\bar b)-2\bar b^0\Big)I\Big]V_x^\varepsilon,V_x^\varepsilon\,\rangle+(\Delta\bar b^0)|V^\varepsilon|^2\Big\}dxdr.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\mathrm{tr}\Big[V_{xx}^\varepsilon\sigma\bar\sigma^0(V_x^\varepsilon)^T\Big]dxdr=\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big(V_{xx}^\varepsilon V_x^\varepsilon\Big)^T\sigma\bar\sigma^0dxdr\\
&=\frac12\,\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\Big[\Big(|V_x^\varepsilon|^2\Big)_x\Big]^T\sigma\bar\sigma^0dxdr=-\frac12\,\mathbb{E}\int_t^T\int_{\mathbb{R}^n}|V_x^\varepsilon|^2\Big(\nabla\cdot[\sigma\bar\sigma^0]\Big)dxdr\\
&\geq -C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}|V_x^\varepsilon|^2dxdr.
\end{aligned}\tag{7.39}
$$

Also, we note that

$$
\mathrm{tr}\Big[V_{xx}^\varepsilon\sigma\sigma_{x_k}^T\Big]=\mathrm{tr}\Big[\Big(V_{xx}^\varepsilon\sigma\sigma_{x_k}^T\Big)^T\Big]=\mathrm{tr}\Big[\sigma_{x_k}\sigma^TV_{xx}^\varepsilon\Big]=\mathrm{tr}\Big[V_{xx}^\varepsilon\sigma_{x_k}\sigma^T\Big].\tag{7.40}
$$

Now, if we denote $\Phi_k=\sigma\sigma_{x_k}^T+\sigma_{x_k}\sigma^T$, then it is $\mathbb{R}^{n\times n}$-valued, symmetric, and

$$
\begin{aligned}
&\mathrm{tr}\Big[V_{xx}^\varepsilon V_{x_k}^\varepsilon\Phi_k\Big]=\mathrm{tr}\Big[\Big(V_{x_k}^\varepsilon\Phi_kV_x^\varepsilon\Big)_x\Big]-\langle\,\nabla\cdot\Big(V_{x_k}^\varepsilon\Phi_k\Big),V_x^\varepsilon\,\rangle\\
&=\nabla\cdot\Big(V_{x_k}^\varepsilon\Phi_kV_x^\varepsilon\Big)-\langle\, V_{x_k}^\varepsilon\nabla\cdot\Phi_k,V_x^\varepsilon\,\rangle-\langle\,\Phi_k(V_x^\varepsilon)_{x_k},V_x^\varepsilon\,\rangle\\
&=\nabla\cdot\Big(V_{x_k}^\varepsilon\Phi_kV_x^\varepsilon\Big)-\langle\, V_{x_k}^\varepsilon\nabla\cdot\Phi_k,V_x^\varepsilon\,\rangle-\frac12\Big[\,\langle\,\Phi_kV_x^\varepsilon,V_x^\varepsilon\,\rangle\Big]_{x_k}+\frac12\,\langle(\Phi_k)_{x_k}V_x^\varepsilon,V_x^\varepsilon\,\rangle\,.
\end{aligned}\tag{7.41}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\mathrm{tr}\Big[V_{xx}^\varepsilon\sum_{k=1}^nV_{x_k}^\varepsilon\sigma\sigma_{x_k}^T\Big]dxdr=\frac12\mathbb{E}\int_t^T\int_{\mathbb{R}^n}\mathrm{tr}\Big[V_{xx}^\varepsilon\sum_{k=1}^nV_{x_k}^\varepsilon\Big(\sigma\sigma_{x_k}^T+\sigma_{x_k}\sigma^T\Big)\Big]dxdr\\
&=\mathbb{E}\int_t^T\!\!\int_{\mathbb{R}^n}\Big\{-\frac12\,\langle\sum_{k=1}^nV_{x_k}^\varepsilon\nabla\cdot[\sigma\sigma_{x_k}^T+\sigma_{x_k}\sigma^T],V_x^\varepsilon\,\rangle+\frac14\,\langle\,\Big(\sum_{k=1}^n[\sigma\sigma_{x_k}^T+\sigma_{x_k}\sigma^T]_{x_k}\Big)V_x^\varepsilon,V_x^\varepsilon\,\rangle\,\Big\}dxdr\\
&\geq -C\mathbb{E}\int_t^T\int_{\mathbb{R}^n}|V_x^\varepsilon|^2dxdr.
\end{aligned}\tag{7.42}
$$

Consequently, making use of (5.18) with $m = 1$, we obtain

$$
\begin{aligned}
-\mathbb{E}\int_{\mathbb{R}^n} |V_x^\varepsilon(t,x)|^2 dx \geq \mathbb{E}\int_t^T \int_{\mathbb{R}^n} \Big\{ & |\sigma^T V_{xx}^\varepsilon + q_x^\varepsilon - \bar{\sigma}^0 (V_x^\varepsilon)^T - \sum_{k=1}^n V_{x_k}^\varepsilon \sigma_{x_k}^T|^2 - |q^\varepsilon|^2 \\
& + 2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{|V_x^\varepsilon|^2}{\varepsilon} - C|V_x^\varepsilon|^2 - C|V^\varepsilon|^2 - |\bar{g}_x|^2 \Big\} dx dr \\
\geq \mathbb{E}\int_t^T \int_{\mathbb{R}^n} \Big\{ |\sigma^T V_{xx}^\varepsilon + q_x^\varepsilon|^2 + 2\psi'\Big(\frac{V^\varepsilon}{\varepsilon}\Big)\frac{|V_x^\varepsilon|^2}{\varepsilon} & - C|V_x^\varepsilon|^2 - |\bar{g}_x|^2 - C|\bar{g}|^2 \Big\} dx dr.
\end{aligned}
$$

By Gronwall's inequality, we obtain (5.22). □