

# A Stochastic Portfolio Optimization Model with Bounded Memory \*

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## Abstract

This paper considers a portfolio management problem of Merton's type in which the risky asset return is related to the return history. The problem is modeled by a stochastic system with delay. The investor's goal is to choose the investment control as well as the consumption control to maximize his total expected, discounted utility. Under certain situations, we derive the explicit solutions in a finite dimensional space.

**Keywords:** Stochastic delay equations, optimal stochastic control, HJB equation

**AMS 2000 subject classifications:** Primary: 34K50; Secondary: 93E20, 49L20

## 1 Introduction.

In this paper, we consider a stochastic portfolio management problem in which the history (delay) of the portfolio performance is taken into consideration. In the classical Merton's model, the price process of the risky asset is modeled by a geometric Brownian motion, which is a Markovian process. In other words, the investor will make his investment decisions only based on the current information (risky asset price, portfolio value, etc.) and will not consider the history of the stock prices or portfolio values. However, in the real world, investors tend to look at the historic performance of the risky asset or their portfolios before they make the investment decision. For example, if the price has been increasing a lot recently, more investors tend to invest more money onto this asset, which will push the price even higher. On the other hand, if the price has been decreasing a lot, more investors tend to sell the asset and invest on other assets, which will drive the price to go down further. To model this phenomenon, we use a stochastic delayed equation to describe the value change of the investor's portfolio, which includes a risky asset and a riskless asset.

In particular, we consider a portfolio which includes a risky asset and a riskless asset. The value of the portfolio follows a stochastic process  $X(t)$  (see (10)) which depends on the following delay

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information:

$$Y(t) = \int_{-h}^0 e^{\lambda\theta} X(t+\theta)d\theta, \quad (1)$$

$$Z(t) = X(t-h), \quad \forall t \in [s, T], \quad (2)$$

where where  $\lambda > 0$  is a constant,  $h > 0$  is the delay parameter and  $s \in [0, T]$  is the initial time.

The initial condition for  $X(t)$  is its historic performance from  $s-h$  to  $s$ :

$$X(t) = \varphi(t-s), \quad \forall t \in [s-h, s].$$

where  $\varphi > 0$  is a continuous function on  $[-h, 0]$ .

Typically, the solution of a stochastic control problem with delay will depend on the initial condition  $\varphi$ , which is in an infinite dimensional space  $C[-h, 0]$ , the space of all continuous functions defined on  $[-h, 0]$ . However, if the system only depends on the delay through processes  $Y(t)$  and  $Z(t)$  processes defined by (1)-(2), it is possible to obtain a solution in a finite dimensional space. Similar stochastic control models have been considered in Elsanousi and Larssen [5], Larssen and Risebro [15]. But they only consider the cases with control in the drift part, not in the diffusion part. The idea of making an assumption like (1)-(2) was used by Kolmanovskii and Maizenberg [12], where a nonsingular stochastic control problem for a certain linear system was considered. In Elsanousi, Øksendal and Sulem [6], a singular stochastic control problem was considered.

We want to point out that stochastic systems with delay in general form  $x_t(s) \equiv x(t+s)$ ,  $s \in [-h, 0]$  were considered in Mohammed [16, 17], Chang, Pang and Pemy [2, 3, 4]. The existence results of the solution in infinite dimensional space were obtained. Due to the infinite dimension, no explicit solution exists, and numerical solutions are very difficult to obtain. This is one motivation for us to study the system with delays in the forms of  $Y(t), Z(t)$  as defined in (1)-(2).

Besides the delayed case, many other modifications of the Merton models have been considered by several researchers. In Bielecki and Pliska [1], Fleming and Sheu [9], continuous time portfolio optimization models of Merton type were considered, where the mean returns of individual asset categories are explicitly affected by underlying economic factors such as dividends and interest rates. The case of a Merton-type model with consumption and the interest rate, which vary in a random, Markovian way, was considered in the second author's other papers (see Fleming and Pang [8], Pang [19, 20]). In some other extension of the model, stochastic volatility is taken into consideration (see Fleming and Hernandez-Hernandez [7], Fouque, Papanicolaou and Sircar [10] and Zariphopoulou [22]).

The paper is organized as follows. In Section 2, we formulate the problem using a stochastic system with delay. In Section 3, we derive the Hamilton-Jacobi-Bellman equation by virtue of dynamic programming principle. A verification theorem is given in Section 4. In Section 5, we consider several interesting examples and compare the results with the classical Merton's solution.

## 2 Problem Formulation.

Consider an investor who can invest his money into a risky asset and a riskless asset. The risky asset can be a stock, a mutual fund, etc. The riskless asset earns a fixed interest rate  $r > 0$ . We can treat the money invested on the riskless asset as money deposited into a bank account. We assume that the investor can freely move his money between two assets at any time and his consumption comes from the riskless asset.

Let  $K(t)$  be the amount invested on the risky asset and  $L(t)$  is the amount invested on the riskless asset. The total wealth is given by  $X(t) = K(t) + L(t)$ . We consider the situation in which

the performance of the risky asset has some memory (delay). Since many investors will look at an asset's past performance before they invest their money on the asset, the increasing investment performance of their wealth in the past tends to drive the investors to invest more on the risky asset, hence it can push the price of the risky asset even higher. On the other hand, if the price has been decreasing a lot, investors tend to sell the asset and invest on other assets, which will drive the price to go down further. To describe this phenomenon, we assume that the performance of the risky asset depends on the following delay variables  $Y(t)$  and  $Z(t)$ :

$$Y(t) \equiv \int_{-h}^0 e^{\lambda\theta} X(t+\theta) d\theta, \quad (3)$$

$$Z(t) \equiv X(t-h), \quad t \in [s, T], \quad (4)$$

where  $\lambda > 0$  is a constant and  $h > 0$  is the delay parameter.  $h$  gives us the duration of the past that the investor usually cares about.

Let  $\{B(t), t \geq 0\}$  be a certain 1-dimensional standard Brownian motion defined on a complete filtered probability space  $(\Omega, \mathcal{F}, P; \mathbf{F})$ , where  $\mathbf{F} = \{\mathcal{F}^t, t \geq 0\}$  is the  $P$ -augmentation natural filtration generated by the Brownian motion  $\{B(t), t \geq 0\}$ .

We assume that  $K(t)$  and  $L(t)$  follow the stochastic differential equations:

$$dK(t) = [(\mu_1 + \mu_2 Y(t) + \mu_3 Z(t))K(t) + I(t)]dt + \sigma K(t)dB(t), \quad (5)$$

$$dL(t) = [rL(t) - C(t) - I(t)]dt, \quad (6)$$

where  $\mu_1, \mu_2, \mu_3, \sigma$  are positive constants,  $I(t)$  is the investment rate on the risky asset at  $t$ , and  $C(t)$  is the consumption rate.

Add them together, and use the fact that  $X(t) = K(t) + L(t)$ , then we get the equation for  $X(t)$ :

$$dX(t) = [(\mu_1 + \mu_2 Y(t) + \mu_3 Z(t))K(t) + rL(t) - C(t)]dt + \sigma K(t)dB(t), \quad \forall t \in [s, T]. \quad (7)$$

The initial condition is the information about  $X(t)$  for  $t \in [s-h, s]$ :

$$X(t) = \varphi(t-s), \quad \forall t \in [s-h, s], \quad (8)$$

where  $\varphi \in \mathbb{J}$  and  $\mathbb{J}$  is defined by  $\mathbb{J} \equiv C[-h, 0]$  is the space for all continuous functions defined on  $[-h, 0]$  equipped with sup-norm:

$$\|\varphi\| = \sup_{\theta \in [-h, 0]} |\varphi(\theta)|. \quad (9)$$

Let  $L^2(\Omega, \mathbb{J})$  be the Banach space of all  $(\mathbf{F}, \text{Borel } \mathbb{J})$ -measurable maps  $\Omega \rightarrow \mathbb{J}$  which are in  $L^2$  in the Bochner sense. For any  $\phi \in L^2(\Omega, \mathbb{J})$ , the Banach norm is given by

$$\|\phi(\omega)\|_2 \equiv \left[ \int_{\Omega} \|\phi(\omega)\|^2 dP(\omega) \right]^{\frac{1}{2}},$$

where the norm  $\|\cdot\|$  is given by (9).

In the equation of  $X(t)$ , we can see that control  $I(t)$  disappears. To describe the allocation between the risky asset and the riskless asset, we now treat  $K(t), C(t)$  as our control variables. The state variables are  $X(t), Y(t)$  and  $Z(t)$ .

As we can see in equation (7), the change of the wealth process  $X(t)$  depends on the delay variables  $Y(t), Z(t)$ . For technical reasons, instead of considering models described by (7) directly, we consider the following slightly modified model:

$$dX(t) = [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + rL(t) - C(t)]dt + \sigma K(t)dB(t), \quad \forall t \in [s, T]. \quad (10)$$

**Remark 2.1** *If we assume that  $K(t) > 0$  almost surely, we can use the following delay variables  $\tilde{Y}(t), \tilde{Z}(t)$ :*

$$\begin{aligned}\tilde{Y}(t) &\equiv \frac{1}{K(t)} \int_{-h}^0 e^{\lambda\theta} X(t+\theta) d\theta, \\ \tilde{Z}(t) &\equiv \frac{X(t-h)}{K(t)}, \quad t \in [s, T],\end{aligned}$$

instead of (3) – (4). Under this assumption, the equation of  $K(t)$ , equation (5) now becomes

$$\begin{aligned}dK(t) &= [(\mu_1 + \mu_2\tilde{Y}(t) + \mu_3\tilde{Z}(t))K(t) + I(t)]dt + \sigma K(t)dB(t) \\ &= [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + I(t)]dt + \sigma K(t)dB(t),\end{aligned}$$

where  $Y(t)$  and  $Z(t)$  are defined as (3) – (4). Then following the similar procedure as we used to derive (7), we will obtain (10) instead of (7).

On the other hand, the above assumption is to describe the following situation. The investor checks his investment amount and the wealth history  $Y(t), Z(t)$ . If  $Y(t)/K(t), Z(t)/K(t)$  are big, which is equivalent to say that with a small amount of investment  $K(t)$ , the wealth return is big, then the investor tends to invest more into the risky asset in order to receive more profit, hence it will push the price up. We only need to guarantee that  $K(t) > 0$ . Later we will see the optimal investment control is positive, provided  $\mu_1 > r$  (see (51)). Therefore, we can request that all admissible control  $k(t) \equiv \frac{K(t)}{X(t)}$  is positive and the model (10) is well-defined.

Further, instead of  $K(t), C(t)$ , we use  $c(t) \equiv \frac{C(t)}{X(t)}, k(t) \equiv \frac{K(t)}{X(t)}$  as our consumption and investment controls, respectively (we will show that  $X(t) > 0$ , a.s. later in Lemma 2.5). It is easy to see that  $L(t) = X(t) - K(t) = X(t)(1 - k(t))$ . Now we can rewrite the equation for  $X(t)$  as

$$\begin{aligned}dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)] dt \\ &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T].\end{aligned}\tag{11}$$

The initial condition is given by

$$X(t) = \varphi(t - s), \quad \forall t \in [s - h, s],\tag{12}$$

where  $\varphi \in \mathbb{J}$  and  $\varphi(\theta) > 0, \forall \theta \in [-h, 0]$ .

**Convention 2.2** *For the rest of the paper, we use the following conventional notation for functional differential equations: If  $\psi \in C([-h, T]; \mathbb{R})$  and  $t \in [0, T]$ , let  $\psi_t \in \mathbb{J}$  be defined by*

$$\psi_t(\theta) = \psi(t + \theta), \quad \forall \theta \in [-h, 0].$$

Since  $\psi_t \in \mathbb{J}$ , its norm is given by the sup-norm:

$$\|\psi_t\| = \sup_{\theta \in [-h, 0]} |\psi_t(\theta)| = \sup_{\theta \in [-h, 0]} |\psi(t + \theta)|.$$

It is easy to see that, by virtue of the above notation, we can rewrite the initial condition (12) as

$$X_s = \varphi.$$

Now we define the admissible control space  $\Pi$  for the control variables  $k(t), c(t)$ :

**Definition 2.3 (Admissible Control Space)** A control policy  $(k(t), c(t))$  is said to be in the admissible control space  $\Pi$  if it satisfies the following conditions:

1.  $(k(t), c(t))$  is  $\mathcal{F}^t$ -measurable for any  $t \in [0, T]$ ;
2.  $c(t) \geq 0, \quad \forall t \in [0, T]$ ;
3. For any  $t \in [0, T]$ , we have

$$|k(t)X(t)| \leq \Lambda_1|X(t) + \mu_3Y(t)|, \quad |c(t)X(t)| \leq \Lambda_2|X(t) + \mu_3Y(t)|,$$

where  $\Lambda_1, \Lambda_2 > 0$  are constants.

We have the following results:

**Lemma 2.4** For any admissible control  $(k(t), c(t)) \in \Pi$ , the system (11) – (12) admits a unique strong solution  $X : [s-h, T] \times \Omega \rightarrow \mathbb{R}$ . In addition, for any  $t \in [s, T]$ ,  $X_t \in \mathbb{J}$ . Furthermore, for any positive integer  $k$ , we have

$$\mathbf{E}\|X_t\|^{2k} \leq C_k[1 + \|\varphi\|^{2k}], \quad \forall t \in [s, T], \quad (13)$$

where  $C_k > 0$  is a constant.

**Proof.** By virtue of

$$\|\varphi\|_2 \leq \|\varphi\|,$$

this lemma is a corollary of Theorem I.2 of Mohammed [17]. □

**Lemma 2.5** The solution  $X(t)$  of the system (11) – (12) satisfies

$$X(t) > 0, \quad \text{almost surely.}$$

**Proof.** Let  $\tilde{X}(t)$  be a solution of

$$d\tilde{X}(t) = [(\mu_1 - r)k(t) - c(t) + r]\tilde{X}(t)dt + \sigma k(t)\tilde{X}(t)dB(t), \quad \forall t \in [s, T], \quad (14)$$

$$\tilde{X}(t) = \varphi(t - s), \quad \forall t \in [s - h, s], \quad (15)$$

for the same  $\varphi > 0$  as in (12). It is easy to verify that its solution is

$$\tilde{X}(t) = \varphi(0) \exp \left\{ \int_s^t \left( (\mu_1 - r)k(\theta) - c(\theta) + r - \frac{1}{2}\sigma^2 k^2(\theta) \right) d\theta + \int_s^t \sigma k(\theta)dB(\theta) \right\}.$$

Apparently, we have

$$\tilde{X}(t) > 0, \quad \text{a.s.}$$

On the other hand, for a given initial condition  $\varphi(\theta) > 0, \forall \theta \in [s - h, s]$ , we must have

$$Y(s) > 0, \quad Z(s) > 0.$$

In addition,  $Y(t), Z(t)$  will remain nonnegative as long as  $X(\theta) \geq 0, \forall \theta \in [t - h, t]$ , and they will reach 0 only if  $X(t)$  has reached 0 before they do.

From the equation (11), we can see that if  $X(t)$  reaches 0 for the first time at  $\hat{t}$ , the diffusion part vanishes and the drift coefficient becomes  $\mu_2Y(\hat{t}) + \mu_3Z(\hat{t})$ , which is positive. Therefore, we can get

$$X(t) \geq 0, \quad \text{a.s.} \quad \forall t \in [s, T].$$

So we can get

$$Y(t) \geq 0, \quad Z(t) \geq 0, \quad a.s. \quad \forall t \in [s, T].$$

Let

$$b(t, x) = ((\mu_1 - r)k(t) - c(t) + r)x + \mu_2 Y(t) + \mu_3 Z(t),$$

be the drift function of (11) and let

$$\tilde{b}(t, x) = ((\mu_1 - r)k(t) - c(t) + r)x,$$

be the drift function of (14). Then, by virtue of  $\mu_2 \geq 0, \mu_3 \geq 0, Y(t) \geq 0$  and  $Z(t) \geq 0$ , it is easy to get that, for any given  $(k(t), c(t))$ , we have

$$b(t, x) \geq \tilde{b}(t, x), \quad \forall x \in \mathbb{R}.$$

Now be virtue of Proposition 2.18 (pp 293) of Karatzas and Shreve [11], we can get that

$$X(t) \geq \tilde{X}(t), \quad a.s. \quad \forall t \in [s, T].$$

Since  $\tilde{X}(t) > 0$ , we have

$$X(t) > 0, \quad a.s. \quad \forall t \in [s, T].$$

□

Let  $U(C)$  be the utility function based on the consumption rate. In addition, assume that  $\Phi$  is the terminal utility function which depends on both  $X(T)$  and  $Y(T)$ . We consider the portfolio optimization problem on a finite time horizon  $[0, T]$  with the objective function

$$J(s, \varphi, k, c) = \mathbf{E}_{s, \varphi} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right],$$

$$\forall (k, c) \in \Pi.$$

Then the value function is given by

$$\begin{aligned} V(s, \varphi) &= \sup_{k, c \in \Pi} J(s, \varphi, k, c) \\ &= \sup_{k, c \in \Pi} \mathbf{E}_{s, \varphi} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right]. \end{aligned} \quad (16)$$

We can see that  $V(s, \varphi)$  is a functional defined on an infinite dimensional space  $[0, T] \times C[-h, 0]$ . In this paper, we will show that under certain conditions,  $V$  can be turned into a function defined on a finite dimensional space through the following way:

$$V(s, \varphi) = V(s, x, y, z),$$

where  $V : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ , and

$$\begin{aligned} x &= x(\varphi) \equiv \varphi(0), \\ y &= y(\varphi) \equiv \int_{-h}^0 e^{\lambda\theta} \varphi(\theta) d\theta, \\ z &= z(\varphi) \equiv \varphi(-h). \end{aligned}$$

Further, we will give the conditions under which the value function  $V$  only depends on  $(s, x, y)$ , i.e.,

$$V(s, \varphi) = V(s, x, y, z) = V(s, x, y). \quad (17)$$

### 3 Hamilton-Jacobi-Bellman Equation.

In this section, we will derive the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function  $V(\varphi)$ , when  $V(\varphi)$  only depends on its initial datum  $\varphi$  through  $x, y$ . In next section, we will derive the condition under which the value function will only depend on  $x, y$ .

Let  $f \in C^{1,2,1}([0, T] \times \mathbb{R}^2)$  and define

$$G(t) = f(s+t, X^\varphi(t), y(X_t^\varphi)),$$

where

$$y(\eta) \equiv \int_{-h}^0 e^{\lambda\theta} \eta(\theta) d\theta, \quad \forall \eta \in \mathbb{J},$$

and

$$X_t(\theta) \equiv X(t+\theta), \quad \forall \theta \in [-h, 0].$$

Then we have the following Ito's formula:

**Lemma 3.1 (Ito's formula)** *Let the system be given by (11)-(12). Then we have*

$$dG(t) = \mathcal{L}f dt + \sigma k x f_x dB(t) + f_y \cdot [x - e^{-\lambda h} z - \lambda y] dt. \quad (18)$$

where

$$\begin{aligned} \mathcal{L}f &= \mathcal{L}^{k,c} f(u, x, y, z) \\ &= f_u + ((\mu_1 - r)k - c + r)x + \mu_2 y + \mu_3 z) f_x + \frac{1}{2} \sigma^2 k^2 x^2 f_{xx}, \end{aligned} \quad (19)$$

and  $\mathcal{L}f(u, x, y, z)$  and the other functions are evaluated at

$$u = s+t, \quad x = X(s), \quad y = y(X_s) = \int_{-h}^0 e^{\lambda\theta} X(s+\theta) d\theta, \quad z = z(X_s) = X(s-h). \quad (20)$$

**Proof.** The idea of the proof is very similar to that of Lemma 2.1 in Elsanousi, Øksendal and Sulem [6]. We repeat it here to make the paper self-contained.

Let the function  $\eta_t(\cdot) \in C[-h, 0]$  denote the path of  $X(t+\theta), \theta \in [-h, 0]$ , then

$$\begin{aligned} \frac{d}{dt} y(\eta_t(\cdot)) &= \frac{d}{dt} \int_{-h}^0 e^{\lambda\theta} \eta_t(\theta) d\theta \\ &= \frac{d}{dt} \left[ e^{\lambda\theta} H(t+\theta) \Big|_{-h}^0 - \int_{-h}^0 \lambda e^{\lambda\theta} H(t+\theta) d\theta \right] \\ &= \frac{d}{dt} \left[ H(t) - e^{-\lambda h} H(t-h) - \lambda \int_{-h}^0 e^{\lambda\theta} H(t+\theta) d\theta \right] \\ &= x - e^{-\lambda h} z - \lambda y, \end{aligned}$$

where  $H$  denotes the antiderivative of  $\eta_t(s)$ . Using the fact that  $G(t) = f(s+t, X^\varphi(t), y(X_t^\varphi))$ , then the result follows from the classical Ito's formula.  $\square$

We assume that the value function  $V$  depends on the initial path  $\varphi$  only through the functionals  $x(\varphi), y(\varphi)$  defined by (20). That is,

$$V(s, \varphi) = V(s, x(\varphi), y(\varphi)) \equiv V(s, x, y). \quad (21)$$

Then we have the following dynamic programming principle:

**Lemma 3.2 (Dynamic programming principle)** Assume that the value function  $V(s, x, y)$  given by (16) and (21) is well defined and assume the system is given by (11)-(12). Then we have

$$V(s, x, y) = \sup_{k, c \in \Pi} \mathbf{E}_{s, \varphi, k, c} \left[ \int_s^t e^{-\beta(\tau-s)} U(c(\tau)X(\tau)) d\tau + e^{-\beta(t-s)} V(t, X(t), Y(t)) \right], \quad (22)$$

for all  $\mathcal{F}^t$ -stopping time  $t \in [s, T]$  and  $(x, y) \in \mathbb{R}^2$ , where  $\varphi \in \mathbb{J}$  is such that  $x = x(\varphi) = X(s)$ ,  $y = y(\varphi) = Y(s)$ .

**Proof.** The proof of this lemma is very similar to that of Theorem 4.2 in Larssen [14], and we omit it here.  $\square$

Using the above lemma and the Ito's formula (see Lemma 3.1), we can prove the following theorem:

**Theorem 3.3 (HJB equation)** Assume that (21) holds and  $V(s, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})$ . Then the value function  $V(s, x, y)$  given by (16) and (21) solves the HJB equation

$$\begin{aligned} \beta V - V_s &= \max_k \left[ \frac{1}{2} (\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x \right] + (r x + \mu_2 y + \mu_3 z) V_x \\ &\quad + \max_{c \geq 0} [-c x V_x + U(c x)] + (x - \lambda y - e^{-\lambda h} z) V_y, \quad \forall z \in \mathbb{R}, \end{aligned} \quad (23)$$

with the boundary condition

$$V(T, x, y) = \Psi(x, y). \quad (24)$$

**Proof.** This is a special case of Theorem 5.1 in Larssen [14]. The proof is omitted here.  $\square$

## 4 The Solution of the HJB Equation.

In this section, we will find an explicit solution of the HJB equation (23) – (24) when the utility function is a HARA type utility function. Further, we will show that the solution is actually the value function we want.

Assume that the utility function is of the HARA type:

$$U(cX) = \frac{1}{\gamma} (cX)^\gamma,$$

where  $\gamma \in (-\infty, 1)$ ,  $\gamma \neq 0$  is a constant. Then we can get

$$\begin{aligned} \beta V - V_s &= \max_k \left[ \frac{1}{2} (\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x \right] + (r x + \mu_2 y + \mu_3 z) V_x \\ &\quad + \max_{c \geq 0} \left[ -c x V_x + \frac{1}{\gamma} (c x)^\gamma \right] + (x - \lambda y - e^{-\lambda h} z) V_y. \end{aligned}$$

The candidate for the optimal control policy is

$$\begin{aligned} k^* &= -\frac{(\mu_1 - r) V_x}{\sigma^2 x V_{xx}}, \\ c^* &= \frac{1}{x} V_x^{\frac{1}{\gamma-1}}. \end{aligned}$$

Plug  $k^*, c^*$  into the HJB equation, and we have

$$\begin{aligned}\beta V - V_s &= -\frac{1}{2} \frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma-1}} \\ &\quad + (rx + \mu_2 y + \mu_3 z) V_x + (x - \lambda y - e^{-\lambda h} z) V_y.\end{aligned}$$

It can be rewritten as

$$\begin{aligned}\beta V - V_s &= -\frac{1}{2} \frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma-1}} + (rx + \mu_2 y) V_x \\ &\quad + (x - \lambda y) V_y + (\mu_3 V_x - e^{-\lambda h} V_y) z.\end{aligned}\tag{25}$$

Assume that the terminal utility function  $\Psi(x, y)$  is given in a form

$$\Psi(x, y) = \frac{1}{\gamma} (x + \mu_3 e^{\lambda h} y)^\gamma.$$

We look for a solution of the form

$$V(s, x, y) = Q(s) \psi(x, y),\tag{26}$$

where  $Q(s)$  and  $\psi(x, y)$  are functions to be determined. From the above equation, it is easy to get that

$$\begin{aligned}V_x &= Q(s) \psi_x(x, y), & V_{xx} &= Q(s) \psi_{xx}(x, y), \\ V_y &= Q(s) \psi_y(x, y), & V_s &= Q'(s) \psi(x, y).\end{aligned}$$

Plug them into the equation (25) for  $V$ , and we can get

$$\begin{aligned}&[\beta Q(s) - Q'(s)] \psi(x, y) \\ &= -\frac{1}{2} \frac{(\mu_1 - r)^2 Q(s) \psi_x^2}{\sigma^2 \psi_{xx}} + \left(\frac{1}{\gamma} - 1\right) [Q(s) \psi_x]^{\frac{\gamma}{\gamma-1}} + (rx + \mu_2 y) Q(s) \psi_x \\ &\quad + (x - \lambda y) Q(s) \psi_y + (\mu_3 \psi_x - e^{-\lambda h} \psi_y) Q(s) z.\end{aligned}\tag{27}$$

Apparently, equation (27) has a solution which does not depend on  $z$  if we have the following condition:

$$(\mu_3 \psi_x - e^{-\lambda h} \psi_y) Q(s) z = 0, \quad \forall z \in \mathbb{R}.\tag{28}$$

Define

$$u \equiv x + \mu_3 e^{\lambda h} y.$$

It can be easily verified that (28) holds if we have

$$\psi(x, y) = g(x + \mu_3 e^{\lambda h} y) = g(u),$$

for a function  $g(\cdot) \in C^2(\mathbb{R})$ . In particular, we assume that

$$g(u) = \frac{1}{\gamma} u^\gamma.$$

Now it is easy to verify that

$$\begin{aligned}\psi_x(x, y) &= g'(u) = u^{\gamma-1}, \\ \psi_{xx}(x, y) &= g''(u) = (\gamma-1)u^{\gamma-2}, \\ \psi_y(x, y) &= \mu_3 e^{\lambda h} g'(u) = \mu_3 e^{\lambda h} u^{\gamma-1}.\end{aligned}$$

Plug them into (27), and we can get

$$\begin{aligned}& \frac{1}{\gamma} [\beta Q(s) - Q'(s)] u^\gamma \\ &= -\frac{1}{2} \frac{(\mu_1 - r)^2}{\sigma^2(\gamma - 1)} Q(s) u^\gamma + \left(\frac{1}{\gamma} - 1\right) [Q(s)]^{\frac{\gamma}{\gamma-1}} u^\gamma \\ & \quad + (rx + \mu_2 y) Q(s) u^{\gamma-1} + (x - \lambda y) \mu_3 e^{\lambda h} Q(s) u^{\gamma-1}.\end{aligned}\tag{29}$$

The above equation can be rewritten as

$$\begin{aligned}& \frac{1}{\gamma} [\beta Q(s) - Q'(s)] u^\gamma \\ &= -\frac{1}{2} \frac{(\mu_1 - r)^2}{\sigma^2(\gamma - 1)} Q(s) u^\gamma + \left(\frac{1}{\gamma} - 1\right) [Q(s)]^{\frac{\gamma}{\gamma-1}} u^\gamma \\ & \quad + [(r + \mu_3 e^{\lambda h})x + (\mu_2 - \lambda \mu_3 e^{\lambda h})y] Q(s) u^{\gamma-1}.\end{aligned}\tag{30}$$

Assume that

$$\mu_2 - \lambda \mu_3 e^{\lambda h} = (r + \mu_3 e^{\lambda h}) \mu_3 e^{\lambda h}.\tag{31}$$

(Some discussions on this assumption are given in Section 5.) Then it is easy to verify that

$$\begin{aligned}& [(r + \mu_3 e^{\lambda h})x + (\mu_2 - \lambda \mu_3 e^{\lambda h})y] Q(s) u^{\gamma-1} \\ &= (r + \mu_3 e^{\lambda h}) Q(s) (x + \mu_3 e^{\lambda h} y) u^{\gamma-1}.\end{aligned}$$

Using the definition of  $u$  ( $u \equiv x + \mu_3 e^{\lambda h} y$ ), we can rewrite the equation (30) as

$$\begin{aligned}& \frac{1}{\gamma} [\beta Q(s) - Q'(s)] u^\gamma \\ &= -\frac{1}{2} \frac{(\mu_1 - r)^2 Q(s) u^\gamma}{\sigma^2(\gamma - 1)} + \left(\frac{1}{\gamma} - 1\right) [Q(s)]^{\frac{\gamma}{\gamma-1}} u^\gamma \\ & \quad + (r + \mu_3 e^{\lambda h}) Q(s) u^\gamma.\end{aligned}\tag{32}$$

Canceling the term  $u^\gamma$  on both sides, we can get

$$\begin{aligned}\frac{1}{\gamma} [\beta Q(s) - Q'(s)] &= -\frac{1}{2} \frac{(\mu_1 - r)^2 Q(s)}{\sigma^2(\gamma - 1)} + \left(\frac{1}{\gamma} - 1\right) [Q(s)]^{\frac{\gamma}{\gamma-1}} \\ & \quad + Q(s)(r + \mu_3 e^{\lambda h}).\end{aligned}\tag{33}$$

The above equation of  $Q(s)$  can be rewritten as

$$Q'(s) = (\gamma - 1) [Q(s)]^{\frac{\gamma}{\gamma-1}} + \left[ \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(r + \mu_3 e^{\lambda h}) \right] Q(s).\tag{34}$$

Define

$$\Lambda \equiv \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(r + \mu_3 e^{\lambda h}). \quad (35)$$

We assume that the all parameters involved here satisfy

$$\Lambda > 0, \quad (36)$$

to guarantee that we have a well-defined solution. We want to point out that the above condition is consistent with a similar condition for classical Merton's problem without delay which is corresponding to the case  $\mu_3 = 0$ . (See Case 1 in Section 5).

Now the equation (34) is equivalent to

$$Q'(s) = (\gamma - 1) [Q(s)]^{\frac{\gamma}{\gamma-1}} + \Lambda Q(s). \quad (37)$$

The above equation can be rewritten as

$$\frac{d}{ds} [e^{-\Lambda s} Q(s)] = (\gamma - 1) e^{\frac{\Lambda s}{\gamma-1}} [e^{-\Lambda s} Q(s)]^{\frac{\gamma}{\gamma-1}}.$$

Or, equivalently,

$$\frac{d}{ds} \left[ (e^{-\Lambda s} Q(s))^{\frac{1}{1-\gamma}} \right] = -e^{\frac{\Lambda s}{\gamma-1}}. \quad (38)$$

On the other hand, at the point  $t = T$ , we have

$$V(T, x, y) = Q(T) \frac{1}{\gamma} (x + \mu_3 e^{\lambda h} y)^\gamma = \Psi(x, y) = \frac{1}{\gamma} (x + \mu_3 e^{\lambda h} y)^\gamma.$$

Therefore, the boundary condition for  $Q(s)$  at  $s = T$  is given by

$$Q(T) = 1. \quad (39)$$

By solving (38) – (39), we can get the solution

$$Q(s) = \left[ \left( 1 - \frac{1-\gamma}{\Lambda} \right) e^{-\frac{\Lambda}{1-\gamma}(T-s)} + \frac{1-\gamma}{\Lambda} \right]^{1-\gamma}. \quad (40)$$

It is easy to verify that, if  $\Lambda > 0$ , we have

$$Q(s) > 0, \quad \forall s \in [0, T].$$

Therefore, the solution of the HJB equation (23) – (24) is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s) (x + \mu_3 e^{\lambda h} y)^\gamma. \quad (41)$$

and the optimal investment ratio and the optimal consumption rate control are

$$k^*(s) = \frac{(\mu_1 - r)(x + \mu_3 e^{\lambda h} y)}{(1 - \gamma)\sigma^2 x}, \quad (42)$$

$$c^*(s) = \frac{x + \mu_3 e^{\lambda h} y}{x} Q(s)^{\frac{1}{\gamma-1}}, \quad (43)$$

where  $Q(s)$  is given by (40) and  $x, y$  are estimated at time  $s$  as the following:

$$x = X(s), \quad y = Y(s) = \int_{-h}^0 e^{\lambda\theta} X(s + \theta) d\theta.$$

Although we have obtained a classical solution of the HJB equation which should be satisfied by the value function, we still need a verification theorem to ensure that the solution defined by (41) is actually equal to the value function defined by (16).

Next we will prove a verification theorem in a more general form.

**Theorem 4.1 (Verification Theorem)** *Let  $X(t)$  be a strong solution of (11)–(12) and  $Y(t), Z(t)$  are given by (1), (2). Assume that  $V(s, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})$  is a solution of the HJB equation (23) – (24) such that*

$$\mathbf{E} \left[ \int_0^T [k(t)X(t)V_x(t, X(t), Y(t))]^2 dt \right] < \infty, \quad \forall k \in \Pi. \quad (44)$$

Then we have

$$V(s, x, y) = \sup_{k, c \in \Pi} \mathbf{E}_{s, \varphi} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right].$$

In addition, if the utility function is given by

$$U(x) = \frac{1}{\gamma} x^\gamma, \quad \gamma \in (-\infty, 1), \gamma \neq 0, \quad (45)$$

then the optimal control policy is given by

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad (46)$$

$$c^* = \frac{1}{x} V_x^{\frac{1}{\gamma-1}}. \quad (47)$$

**Proof.** Using the notation  $\mathcal{L}^{k,c}$  defined by (19), we can rewrite the equation (23) as

$$\max_{k, c \geq 0} [\mathcal{L}^{k,c} V(s, x, y) + U(cx)] - \beta V(s, x, y) = 0. \quad (48)$$

Let  $V(s, x, y)$  be a solution of the equation (48). For any given admissible control  $(k, c) \in \Pi$  and for any  $(s, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ , we must have

$$\beta V(s, x, y) - \mathcal{L}^{k,c} V(s, x, y) \geq U(cx). \quad (49)$$

On the other hand, applying the Ito's formula (18) to  $V(t, X(t), Y(t))$ , we can get

$$\begin{aligned} & d[e^{-\beta t} V(t, X(t), Y(t))] \\ &= e^{-\beta t} [-\beta V(t, X(t), Y(t)) dt + dV(t, X(t), Y(t))] \\ &= e^{-\beta t} \left[ (-\beta V(t, X(t), Y(t)) + \mathcal{L}^{k,c} V(t, X(t), Y(t))) dt \right. \\ & \quad \left. + \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \right]. \end{aligned}$$

Integrating it from  $s$  to  $T$ , and using (49), we can get

$$\begin{aligned}
& e^{-\beta T}V(T, X(T), Y(T)) - e^{-\beta s}V(s, x, y) \\
&= \int_s^T e^{-\beta t} \left( -\beta V(t, X(t), Y(t)) + \mathcal{L}^{k,c}V(t, X(t), Y(t)) \right) dt \\
&\quad + \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
&\leq - \int_s^T e^{-\beta t} U(c(t)X(t)) dt + \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t).
\end{aligned}$$

Therefore, by virtue of the boundary condition (24), we can get

$$\begin{aligned}
V(s, x, y) &\geq e^{-\beta(T-s)}V(T, X(T), Y(T)) + \int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \\
&\quad - \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
&= e^{-\beta(T-s)}\Psi(X(T), Y(T)) + \int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \\
&\quad - \int_s^T e^{-\beta(t-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t).
\end{aligned}$$

Using the condition (44), it is easy to obtain that

$$\int_s^T e^{-\beta(t-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t)$$

is a martingale. Therefore, by taking expectations for both sides, we can get

$$\begin{aligned}
V(s, x, y) &\geq \mathbf{E}_{s,\varphi,k,c} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt \right. \\
&\quad \left. + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right].
\end{aligned}$$

Since it holds for all  $(k, c) \in \Pi$ , we must have

$$\begin{aligned}
V(s, x, y) &\geq \sup_{k,c \geq 0} \mathbf{E}_{s,\varphi,k,c} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt \right. \\
&\quad \left. + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right].
\end{aligned}$$

On the other hand, when the utility function is given by (45), and if we take  $(k, c) = (k^*, c^*)$  as defined in (46), then all the above inequalities can be replaced by equalities. In other words, we can have

$$\begin{aligned}
V(s, x, y) &= \mathbf{E}_{s,\varphi,k^*,c^*} \left[ \int_s^T e^{-\beta(t-s)} U(c^*(t)X(t)) dt \right. \\
&\quad \left. + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right].
\end{aligned}$$

Now the proof is complete.  $\square$

Now we can show that the function defined by (41) is a classical solution of (23) – (24) and the optimal control policy is given by (42) – (43).

Let  $X(t)$  be a strong solution of (11) – (12) and  $Y(t), Z(t)$  are given by (1), (2).

**Theorem 4.2** *Assume that the utility function is given by*

$$U(x) = \frac{1}{\gamma}x^\gamma, \quad \gamma \in (0, 1),$$

and the terminal utility function is give by

$$\Psi(x, y) = \frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^\gamma.$$

In addition, assume that condition (31) holds. Then the function  $V(s, x, y)$  given by (41) is a classical solution of the HJB equation (23) – (24) and it is equal to the value function defined by (16) – (17), that is

$$V(s, x, y) = \sup_{k, c \in \Pi} \mathbf{E}_{s, \varphi} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right]. \quad (50)$$

In addition, the optimal control policy is give by

$$k^*(t) = \frac{(\mu_1 - r)(X(t) + \mu_3 e^{\lambda h} Y(t))}{(1 - \gamma)\sigma^2 X(t)}, \quad (51)$$

$$c^*(t) = \frac{X(t) + \mu_3 e^{\lambda h} Y(t)}{X(t)} Q(t)^{\frac{1}{\gamma-1}}, \quad \forall t \in [s, T], \quad (52)$$

where  $Q(\cdot)$  is defined by (40).

**Proof.** From the derivation of  $V(s, x, y)$ , it is easy to see that  $V(s, x, y)$  given by (41) is a classical solution of the HJB equation (23) – (24). To use Theorem 4.1, we need to verify that condition (44) is satisfied.

Using (41), we can get

$$V_x(t, X(t), Y(t)) = Q(t)(X(t) + \mu_3 e^{\lambda h} Y(t))^{\gamma-1}.$$

Using  $Q(t) > 0$ , then we have

$$|V_x(t, X(t), Y(t))| = Q(t)|X(t) + \mu_3 e^{\lambda h} Y(t)|^{\gamma-1}.$$

On the other hand, by the definition of the admissible control space  $\Pi$ , we can get that

$$\begin{aligned} |k(t)X(t)| &\leq \Lambda_1 |X(t) + \mu_3 Y(t)| \\ &\leq \Lambda_1 |X(t) + \mu_3 e^{\lambda h} Y(t)|. \end{aligned}$$

By the definition of  $Y(t)$ , we have

$$\begin{aligned} Y(t) &= \int_{-h}^0 e^{\lambda \theta} X(t + \theta) d\theta \\ &\leq \int_{-h}^0 X(t + \theta) d\theta \\ &\leq h \max_{\theta \in [-h, 0]} |X(t + \theta)| \\ &= h \|X_t\|. \end{aligned}$$

In addition, it is easy to see that

$$|X(t)| \leq \|X_t\|.$$

Therefore, noting that  $Q(t)$  is bounded on  $[0, T]$ , we can get

$$\begin{aligned} |k(t)X(t)V_x(t, X(t), Y(t))| &\leq \Lambda_1 Q(t)|X(t) + \mu_3 e^{\lambda h} Y(t)|^\gamma \\ &\leq \Lambda_3 \|X_t\|^\gamma, \end{aligned}$$

where  $\Lambda_3 > 0$  is a constant that does not depend on  $t$ .

Therefore, by virtue of (13) and noting that  $2\gamma < 2$  we can get

$$\begin{aligned} &\mathbf{E} \left[ \int_0^T [k(t)X(t)V_x(t, X(t), Y(t))]^2 dt \right] \\ &\leq \mathbf{E} \left[ \int_0^T \Lambda_3^2 \|X_t\|^{2\gamma} dt \right] \\ &\leq \mathbf{E} \left[ \int_0^T \Lambda_3^2 (1 + \|X_t\|)^{2\gamma} dt \right] \\ &\leq \mathbf{E} \left[ \int_0^T \Lambda_3^2 (1 + \|X_t\|)^2 dt \right] \\ &\leq \mathbf{E} \left[ \int_0^T \Lambda_3^2 (2 + 2\|X_t\|^2) dt \right] \\ &= \Lambda_3^2 \int_0^T \mathbf{E}[2 + 2\|X_t\|^2] dt \\ &\leq \Lambda_3^2 \left( 2T + \int_0^T 2C_2(1 + \|\varphi\|^2) dt \right) \\ &< \infty, \end{aligned}$$

where  $\Lambda_3 > 0$  is a constant that does not depend on  $t$ . Therefore, the condition (44) is satisfied.

In addition, it is not hard to show that  $(k^*, c^*) \in \Pi$ . This completes the proof.  $\square$

## 5 Some Special Cases.

In this section, we discuss some examples. For convenience, we rewrite the dynamic equation for the wealth process  $X(t)$  here:

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)] dt \\ &\quad + [\sigma k(t)X(t)] dB(t), \quad \forall t \in [s, T]. \end{aligned} \tag{53}$$

The initial condition is given by

$$X(s + \theta) = \varphi(\theta), \quad \forall \theta \in [-h, 0]. \tag{54}$$

In last section, we have obtained an explicit solution given the assumption (equation (31)):

$$\mu_2 - \lambda \mu_3 e^{\lambda h} = (r + \mu_3 e^{\lambda h}) \mu_3 e^{\lambda h}. \tag{55}$$

We discuss some interesting cases here.

**Case 1.** Let us start with a simple case by taking  $\mu_3 = 0$ . It is easy to see that if (55) is satisfied, we must have  $\mu_2 = 0$ . In this case, the dynamic equation of  $X(t)$  does not depend on  $Y(t)$  or  $Z(t)$  explicitly. Now the equations for  $X(t)$  are

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t)] dt \\ &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T], \\ X(s + \theta) &= \varphi(\theta), \quad \forall \theta \in [-h, 0]. \end{aligned} \tag{56}$$

The optimal control policy is

$$\begin{aligned} k_s^* &= \frac{(\mu_1 - r)}{\sigma^2(1 - \gamma)}, \\ c_s^* &= [Q(s)]^{\frac{1}{\gamma-1}}, \end{aligned}$$

where  $Q(s)$  is given by

$$Q(s) = \left[ \left(1 - \frac{1 - \gamma}{\Lambda}\right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1 - \gamma}{\Lambda} \right]^{1-\gamma}$$

and  $\Lambda$  is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma r.$$

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s) x^\gamma.$$

Actually, it is easy to verify that this value function and the optimal investment and consumption control polices are the same with those of the classical Merton's problem on a finite time horizon with objective function

$$J(s, x) = \max_{k, c} \mathbf{E}_{s, x} \left[ \int_s^T e^{-\beta(t-s)} \frac{1}{\gamma} (c(t)X(t))^\gamma dt + e^{-\beta(T-s)} \frac{1}{\gamma} [X(T)]^\gamma \right],$$

with dynamic equations for  $X(t)$  being

$$\begin{aligned} dX(t) &= [(\mu_1 - r)k(t) - c(t) + r]X(t)dt + \sigma k(t)X(t)dB(t), \\ X(0) &= x. \end{aligned} \tag{58}$$

where  $x = x(\varphi) = \varphi(0)$ . A condition to ensure the existence of the solution is

$$\beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma r > 0,$$

which is consistent with the condition  $\Lambda > 0$  (condition (36)) when  $\mu_3 = 0$ .

**Case 2.** Let  $\mu_3 = \nu e^{-\lambda h}$  for a constant  $\nu > 0$  and let  $\mu_2 = \nu^2 + \nu(r + \lambda)$ . It is easy to verify that (55) holds. Now the equations (56 – 57) of  $X(t)$  become

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + (\nu^2 + \nu(r + \lambda))Y(t) \\ &\quad + \nu e^{-\lambda h} Z(t)]dt + \sigma k(t)X(t)dB(t), \text{ nonumber} \\ X(s + \theta) &= \varphi(\theta), \quad \forall \theta \in [-h, 0]. \end{aligned} \tag{59}$$

The optimal control is now given by

$$k_s^* = \frac{(\mu_1 - r)(x + \nu y)}{(1 - \gamma)\sigma^2 x} \quad (60)$$

$$= \frac{\mu_1 - r}{\sigma^2(1 - \gamma)} + \left[ \frac{(\mu_1 - r)\nu}{\sigma^2(1 - \gamma)} \right] \frac{y}{x};$$

$$c_s^* = \frac{x + \nu y}{x} Q(s)^{\frac{1}{\gamma-1}} \quad (61)$$

$$= \left( 1 + \nu \frac{y}{x} \right) Q(s)^{\frac{1}{\gamma-1}},$$

where  $Q(s)$  is given by

$$Q(s) = \left[ \left( 1 - \frac{1 - \gamma}{\Lambda} \right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1 - \gamma}{\Lambda} \right]^{1-\gamma}$$

and  $\Lambda$  is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(\nu + r).$$

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s)(x + \nu y)^\gamma.$$

As we can see, both the control policy and the value function now depend on the parameter  $\nu$ .

**Case 3.** Now we assume  $\mu_3 = e^{-\lambda h}$ . Then we have

$$r + \mu_3 e^{\lambda h} = r + 1.$$

If  $\mu_2$  satisfies

$$\mu_2 = r + 1 + \lambda,$$

then it is easy to verify that (55) holds. Then the dynamic equation for  $X(t)$  is now given by

$$dX(t) = \left[ ((\mu_1 - r)k(t) - c(t) + r)X(t) + (\lambda + r + 1)Y(t) + e^{-\lambda h}Z(t) \right] dt$$

$$+ \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T], \quad (62)$$

$$X(s + \theta) = \varphi(\theta), \quad \forall \theta \in [-h, 0].$$

The optimal control policy is

$$k_s^* = \frac{(\mu_1 - r)(x + y)}{\sigma^2(1 - \gamma)x}$$

$$= \frac{\mu_1 - r}{\sigma^2(1 - \gamma)} + \left[ \frac{(\mu_1 - r)}{\sigma^2(1 - \gamma)} \right] \frac{y}{x},$$

$$c_s^* = \frac{x + y}{x} Q(s)^{\frac{1}{\gamma-1}} \quad (63)$$

$$= \left( 1 + \frac{y}{x} \right) Q(s)^{\frac{1}{\gamma-1}},$$

where  $Q(s)$  is given by

$$Q(s) = \left[ \left( 1 - \frac{1 - \gamma}{\Lambda} \right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1 - \gamma}{\Lambda} \right]^{1-\gamma}$$

and  $\Lambda$  is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(1 + r).$$

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s)(x + y)^\gamma.$$

As we can see, both the control policy and the value function now depend on the delay variable  $y$ . Actually, this is a special case of Case 2 with  $\nu = 1$ .

**Final Remark.** From the condition (55), we can get

$$\begin{aligned} \mu_2 &= \lambda \mu_3 e^{\lambda h} + (r + \mu_3 e^{\lambda h}) \mu_3 e^{\lambda h} \\ &= \mu_3 e^{\lambda h} (\lambda + r + \mu_3 e^{\lambda h}). \end{aligned}$$

So it is easy to see that  $\mu_2 = 0$  if and only if  $\mu_3 = 0$ , provided that  $\mu_3 \geq 0$ , and

$$\lim_{\mu_3 \rightarrow \infty} \mu_2 = \infty.$$

In other words, the price change of  $X(t)$  must depend on both  $Y(t)$  and  $Z(t)$  at the same time with similar manner in order to obtain an explicit solution  $V(s, x, y)$ . Even though it is a relatively simple model, to obtain a solution in finite dimensional space is not easy. This indicates that stochastic systems with delay are very difficult to deal with.

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