Viscosity Solutions of Infinite Dimensional Black-Scholes Equation and Numerical Approximations*

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Abstract

This paper addresses the issues of the viscosity solution and its finite difference approximation for an infinite dimensional Black-Scholes equation obtained from Chang and Youree [5]. The equation arises from a consideration of an European option pricing problem in a (B, S)-market in which the stock price and the asset in the riskless bank account both have hereditary structures. Under a general condition on the payoff function of the option, it is shown that the pricing function is the unique viscosity solution of the infinite dimensional Black-Scholes equation. In addition, a finite difference approximation of the viscosity solution is also provided.

Keywords: Black-Scholes equation, stochastic functional differential equations, viscosity solutions, numerical methods.

AMS 2000 subject classifications: primary 34K28, 91B28; secondary 34K50, 60H35.

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1 Introduction

A European contingent claim (or option) is a contract giving the buyer of the contract (or simply the buyer) the right to buy from or sell to the contract writer (or simply writer) a share of a particular stock at a pre-determined price $K > 0$ (called the strike price) and at a pre-determined time $T > 0$ (called the expiration date) in the future. The right to buy (respectively, to sell) a share of the stock is called a *call* (respectively, a *put*) option. The European (call or put) option pricing problem is, briefly, to determine the fee (called the *rational price*) that the writer should receive from the buyer for the rights of the contract and also to determine the trading strategy the writer should use to invest this fee in the (B, S)-market in such a way as to ensure that the writer will be able to cover the option if it is exercised. The fee should be large enough that the writer can, with riskless investing, cover the option, but be small enough that the writer does not make an unfair (*i.e.,* riskless) profit.

The pricing of (European) options in the continuous-time (B, S)-market has been a subject of extensive research in recent years. Explicit results for the European call option obtained (see *e.g.,* Black & Scholes [3], Harrison & Kreps [10], Harrison & Pliska [11], Merton [17, 18], Shiryaev et al [21]) for the idealized Black-Scholes (B, S)-market often consists of the a simple interest savings account $\{B(t), t \geq 0\}$ and a stock whose price process $\{S(t), t \geq 0\}$ satisfy geometric Brownian motion process. In order to better model real-life options, various generalizations of the model have been made. In particular, (B, S)-markets that have hereditary structure or delayed responses (Chang & Youree [4, 5], Arriojas et al [1], Kazmerchuk et al [13, 14, 15]) in which various versions of stochastic functional differential equations were introduced to describe the dynamics of the bank account and the stock price.

This paper addresses the issues of the viscosity solution and its finite difference approximation for an infinite dimensional Black-Scholes equation obtained from Chang and Youree [5]. The infinite dimensional Black-Scholes equation arises from the (B, S)-market in which the savings account $\{B(t), \quad t \geq -h\}$ grows according to the following linear (deterministic) functional differential equation

$$dB(t) = L(B_t)dt \tag{1}$$

and the stock price process $\{S(t), t \geq -h\}$ satisfies the following nonlinear stochastic functional differential equation

$$\frac{dS(t)}{S(t)} = f(S_t)dt + g(S_t)dW(t), \; t \geq 0, \tag{2}$$

where

$$L(B_t) \equiv \int_{-h}^{0} B(t+\theta)d\eta(\theta)dt, \; t \geq 0,$$

$\eta : [-h, 0] \to \mathbf{R}$ is of bounded variation, and $h > 0$ represents the time delay or the duration of a finite memory of the market. In addition to the nonlinearity of Equation (2), one distinct feature is that the stock appreciation $f(S_t)$ and stock volatility $g(S_t)$ at time $t \geq 0$ are given nonlinear functions of the stock prices S_t (see Definition 2.1) over the time interval $[t-h, t]$ instead of just the stock price $S(t)$ at time t only. The distinction between $S(t)$ and S_t shall be made clear in Section 2 below.

It is clear that when $h = 0$, Equation (2) reduces to the following nonlinear stochastic ordinary differential equation

$$\frac{dS(t)}{S(t)} = f(S(t))dt + g(S(t))dW(t), \; t \geq 0, \tag{3}$$

of which the Black-Scholes (B, S)-market is a special case. When $h > 0$, Equation (2) is general enough to include the linear model considered by Chang & Youree [4], and pure discrete delay models considered by Arriojas et al [1], and Kazmerchuk et al [12], [13], and [14].

Under the (B, S)-market described by equations (1) and (2), and with a general payoff function $\lambda(S_T)$ at expiration time T, Chang and Youree [5] derived an infinite-dimensional Black-Scholes equation (see Equations (12) and (13) in this paper) for the pricing function for the European options. This generalized Black-Scholes equation involves extended Fréchet derivatives of functions on the function space $\mathbf{C}$ (see Section 3.1 for definitions). Under certain sufficiently smooth conditions, it was shown that the pricing function is a classical solution of the generalized Black-Scholes equation. In there, an algorithm for computing the solution of the infinite-dimensional Black-Scholes equation is also obtained via a double sequence of polynomials of a certain bounded linear functional on a Banach space and the time variable.

This paper shows, under a general payoff function, that the pricing function is the unique viscosity solution of the infinite dimensional Black-Scholes equation obtained in Chang and Youree [5]. In addition, a finite difference approximation result based on the result obtained by Barles and Souganidis [2] is also provided in this paper.

This paper is organized as follows. Section 2 summarizes the definitions and key ingredients that will be used throughout this paper. The concepts of Fréchet derivative and extended Fréchet derivative are introduced in Section 3, along with results needed to make use of these derivatives. This section also state the infinite-dimensional Black-Scholes equation obtained in Chang and Youree [5] under the differentiability assumption of the pricing function. Section 4 shows that the pricing function is the unique viscosity solution of the infinite dimensional Black-Scholes equation. A finite difference approximation result in obtained in Section 5.

2 Problem Formulation

Let $h > 0$ be a fixed constant, and let $J = [-h, 0]$ denote the duration of the bounded memory of the stochastic functional differential equations considered in this paper. For the sake of simplicity, denote $C(J;\mathbb{R})$, the space of continuous functions $\phi : J \to \mathbb{R}$, by $\mathbf{C}$. Note that $\mathbf{C}$ is a real separable Banach space under the sup-norm defined by

$$\|\phi\| = \sup_{t\in J} |\phi(t)|, \quad \phi \in \mathbf{C} \tag{4}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}$.

We denote by $(\cdot\,|\,\cdot)$ the inner product in $L^2(J,\mathbb{R})$ as the following

$$(\phi|\psi) = \int_{-h}^{0} \langle \phi(s), \psi(s)\rangle ds, \qquad \forall \phi, \psi \in \mathbf{C},$$

where $\langle\cdot\,,\,\cdot\rangle$ is the inner product in $\mathbb{R}$. In addition, we define

$$\|\phi\|_2 = (\phi|\phi)^{\frac{1}{2}}.$$

Note that the space $\mathbf{C}$ can be continuously embedded into $L^2(J;\mathbb{R})$.

Let $L^2(\Omega, \mathbf{C})$ be the space of $\mathbf{C}$-valued random variables $\Theta : \Omega \to \mathbf{C}$ such that

$$\|\Theta\|_{L^2} = \left\{ \int_\Omega \|\Theta(\omega)\|^2 dP(\omega) \right\}^{\frac{1}{2}} < \infty.$$

Let $L^2(\Omega, \mathbf{C}; \mathcal{F}(t))$ be those $\Theta \in L^2(\Omega, \mathbf{C})$ which are $\mathcal{F}(t)$-measurable.

Convention (2.1). *Throughout the end, we use the following conventional notation for functional differential equations (see Hale [9]):*

If $\psi \in C([-h, \infty); \mathbb{R})$ and $t \in \mathbb{R}_+$, let $\psi_t \in \mathbf{C}$ be defined by $\psi_t(\theta) = \psi(t + \theta)$, $\theta \in J$.

Let $\{W(t), t \geq 0\}$ be an m-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}(t), t \geq 0\}$ is the P-augmentation of the natural filtration $\{\mathcal{F}^W(t), t \geq 0\}$ generated by the Brownian motion $\{W(t), t \geq 0\}$, *i.e.,* if $t \geq 0$,

$$\mathcal{F}^W(t) = \sigma\{W(s), 0 \leq s \leq t\}$$

and

$$\mathcal{F}(t) = \mathcal{F}^W(t) \vee \{A \subset \Omega | A \subset B, B \in \mathcal{F}, P(B) = 0\}$$

where the operator $\vee$ denotes that $\mathcal{F}(t)$ is the smallest σ-algebra such that $\mathcal{F}^W(t) \subset \mathcal{F}(t)$ and $\{A \subset \Omega | A \subset B, B \in \mathcal{F}, P(B) = 0\} \subset \mathcal{F}(t)$.

Let T and t be positive numbers, $T > 0$ denotes the expiration of the European option. We assume that the stock price process $\{S(t), t \in [-h, T]\}$ satisfies the following nonlinear stochastic functional differential equation: stochastic functional differential equations:

$$\frac{dS(t)}{S(t)} = f(S_t)dt + g(S_t)dW(t), \quad 0 \leq t \leq T; \tag{5}$$

with the initial price function ψ, where ψ is a given $\mathbf{C}$-valued random variable that is $\mathcal{F}(t)$-measurable. The functions, $f : \mathbf{C} \to \mathbb{R}$ and $g : \mathbf{C} \to \mathbb{R}$ are given deterministic functions.

Definition 2.1 *A process $\{S(s; \psi), s \in [-h, T]\}$ is said to be a (strong) solution of (5) on the interval $[-h, T]$ and through the initial price function $\psi \in \mathbf{C}$ if it satisfies the following conditions:*

1. *$S(t) = \psi(t) \quad$ for all $t \in [-h, 0]$;*
2. *$S(u; \psi)$ is $\mathcal{F}(u)$-measurable for each $u \in [0, T]$;*

3. *The process* $\{S(u;\psi), u \in [0,T]\}$ *is continuous and satisfies the following stochastic integral equation* P*-a.s.*

$$S(u) = \psi(0) + \int_0^u S(\tau)f(S_\tau)d\tau + \int_0^u S(\tau)g(S_\tau)dW(\tau),\ 0 \le u \le T. \quad (6)$$

In addition, the solution process $\{S(u;\psi), u \in [-h,T]\}$ *is said to be (strongly) unique if* $\{\tilde{S}(u;\psi), u \in [-h,T]\}$ *is also a solution of (5) on* $[-h,T]$ *and through the same initial datum* ψ*, then*

$$P\{S(u;\psi) = \tilde{S}(u;\psi), \forall u \in [0,T]\} = 1.$$

Throughout the end, we assume that the functions $f : \mathbf{C} \to \mathbb{R}$, and $g : \mathbf{C} \to \mathbb{R}$ satisfy the following conditions to ensure the existence and uniqueness of a global (strong) solution

$$\{S(u;\psi), u \in [-h,T]\}$$

for each $\psi \in L^2(\Omega, C; F(\alpha))$. (See Mohammed [19, 20].) The functions f and g are continuous with at most linear growth and the following linear growth and Lipschitz conditions.

Assumption 2.2 *There exists a constant* $\Lambda > 0$ *such that*

$$|\varphi(0)f(\varphi) - \varphi(0)f(\phi)| + |\varphi(0)g(\varphi) - \phi(0)g(\phi)| \le \Lambda\|\varphi - \phi\|,\ \forall \varphi, \phi \in \mathbf{C}.$$

Assumption 2.3 *There exists a constant* $K > 0$ *such that*

$$|\phi(0)f(\phi)| + |\phi(0)g(\phi)| \le K(1 + \|\phi\|),\ \forall \phi \in \mathbf{C}.$$

Note that the $\mathbf{C}$-valued process $\{S_u(\psi), u \in [0,T]\}$ is a Markov process (see Mohammed [19], [20]).

In addition to the stock price precess, we also assume the rickless tradable asset, $\{B(t),\ \ t \ge -h\}$ has some memory. More particular, we assume that $\{B(t),\ \ t \ge -h\}$ grows according to the following linear (deterministic) functional differential equation

$$dB(t) = L(B_t)dt \quad (7)$$

where

$$L(B_t) \equiv \int_{-h}^0 B(t+\theta)d\eta(\theta)dt,\ t \ge 0, \quad (8)$$

and $\eta : [-h, 0] \rightarrow \mathbf{R}$ is of bounded variation. Given the above η, we can define the discount rate $r > 0$ such that

$$r = \int_{-h}^{0} e^{r\theta} d\eta(\theta). \tag{9}$$

Now consider the non-arbitrage price of a European type option. Given a payoff function $\lambda : \mathbf{C} \rightarrow \mathbb{R}_{+}$, formally the price function $V : [0,T] \times \mathbf{C} \rightarrow \mathbb{R}$ is defined as follows:

$$V(t,\psi) \quad = \quad \widetilde{\mathbf{E}}\left[e^{-r(T-t)}\lambda(S_T) \middle| S_t = \psi \right] \tag{10}$$

where $\widetilde{\mathbf{E}}$ represents the expectation for a suitable probability measure $\tilde{P}$ defined in Lemma (2.8) of Chang and Youree [4]. Given a continuous payoff λ with a linear growth, one can show that the pricing function V is also a continuous function with linear growth.

The reward function $\lambda : \mathbf{C} \rightarrow \mathbf{R}$ can be a very general measurable function. In particular, it can includes the following as special cases.

Example 2.4 *(i)* $\lambda(\varphi) = (\varphi(0) - K)^{+}$. *This is the reward function for a standard European call option. Note that this is also the payoff function considered in Arriojas et al [1].*

(ii) $\lambda(\varphi) = (K - \varphi(0))^{+}$. *This is the reward function for a standard European put option.*

(iii) $\lambda(\varphi) = \int_{-h}^{0} H(\varphi(\theta)e^{-r\theta}, \varphi(0))d\theta$ *for some function* $H : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$. *This is the payoff function for the European option considered in Kazmerchuk et al [13, 14, 15]. Note that the reward function* $\lambda \in C^2(\mathbf{C})$ *if the function* $H \in C^{2,2}(\mathbf{R} \times \mathbf{R})$ *(i.e.,* $H(x,y)$ *is twice continuously differentiable with respect to each of its two variables).*

3 Infinite-Dimensional Black-Scholes Equation

3.1 The Infinitesimal Generator

Let $\mathbf{C}^*$ and $\mathbf{C}^\dagger$ be the space of bounded linear functionals $\Phi : \mathbf{C} \rightarrow \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : \mathbf{C} \times \mathbf{C} \rightarrow \mathbb{R}$, of the space $\mathbf{C}$, respectively.

They are equipped with the operator norms which will be, respectively, denoted by $\|\cdot\|^*$ and $\|\cdot\|^\dagger$.

Let $\mathbf{B} = \{v\mathbf{1}_{\{0\}}, v \in \mathbb{R}^n\}$, where $\mathbf{1}_{\{0\}} : [-h, 0] \to \mathbb{R}$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-h, 0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$\mathbf{C} \oplus \mathbf{B} = \{\phi + v\mathbf{1}_{\{0\}} \mid \phi \in \mathbf{C}, v \in \mathbb{R}^n\}$$

and equip it with the norm $\|\cdot\|$ defined by

$$\|\phi + v\mathbf{1}_{\{0\}}\| = \sup_{\theta \in [-h,0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbf{C}, v \in \mathbb{R}^n.$$

Note that for each sufficiently smooth function $\Phi : \mathbf{C} \to \mathbb{R}$, its first order Fréchet derivative (with respect to $\phi \in \mathbf{C}$), $D\Phi(\varphi) \in \mathbf{C}^*$, has a unique and continuous linear extension $\overline{D\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi) \in \mathbf{C}^\dagger$, has the unique and continuous linear extension $\overline{D^2\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^\dagger$, where $(\mathbf{C} \oplus \mathbf{B})^*$and $(\mathbf{C} \oplus \mathbf{B})^\dagger$ are spaces of bounded linear and bilinear functionals of $\mathbf{C} \oplus \mathbf{B}$, respectively. (See Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [19] for details).

For a Borel measurable function $\Phi : \mathbf{C} \to \mathbb{R}$, we also define

$$\mathcal{S}(\Phi)(\phi) = \lim_{t \to 0+} \frac{1}{t} \left[\Phi(\tilde{\phi}_t) - \Phi(\phi) \right]$$

for all $\phi \in \mathbf{C}$, where $\tilde{\phi} : [-h, T] \to \mathbb{R}^n$ is an extension of ϕ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-h, 0) \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and again $\tilde{\phi}_t \in \mathbf{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-h, 0].$$

Let $\mathcal{D}(\mathcal{S})$, the domain of the operator $\mathcal{S}$, be the set of $\Phi : \mathbf{C} \to \mathbb{R}$ such that the above limit exists for each $\phi \in \mathbf{C}$.

Throughout the end, let $C^{1,2}_{lip}([0,T]\times\mathbf{C})$ be the space of functions $\Phi : [0,T]\times\mathbf{C}\to\mathbb{R}$ such that $\frac{\partial\Phi}{\partial t} : [0,T]\times\mathbf{C}\to\mathbb{R}$ and $D^2\Phi : [0,T]\times\mathbf{C}\to\mathbf{C}^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t,\phi)-D^2\Phi(t,\varphi)\|^\dagger \le K\|\phi-\varphi\| \quad \forall t\in[0,T],\ \phi,\varphi\in\mathbf{C}.$$

By changing probability measure to the risk-neutral probability measure, the equation (2) can be replaced by

$$dS(t) = S(t)[rdt + g(S_t)d\widetilde{W}(t)], \tag{11}$$

where $\widetilde{W}(t)$ is a standard Brownian motion under the new probability measure $\tilde{P}$. Therefore, in the sequel, we will restrict our attention to the above price process instead of (2). The details of the probability measure change from P to $\tilde{P}$ can be found in Chang and Youree [5]. In addition, we have the following result from Chang and Youree [5]:

Theorem 3.1 *Let $V(t,\phi) = e^{-r(T-t)}\widetilde{\mathbf{E}}[\lambda(S_T)|S_t=\phi]$, where $\lambda(\cdot)$ is the reward function. Suppose that $V(t,\phi)\in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathcal{D}(\mathcal{S})$. Then for any $(t,\phi)\in[0,T)\times\mathbf{C}_+$, we have*

$$\begin{aligned} rV(t,\phi) &= \frac{\partial V(t,\phi)}{\partial t} + \mathcal{S}(V)(t,\phi) + \overline{DV(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}}) \\ &\quad + \frac{1}{2}\overline{D^2V(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}},\phi(0)g(\phi)\mathbf{1}_{\{0\}}) \end{aligned} \tag{12}$$

with

$$V(T,\phi) = \lambda(\phi). \tag{13}$$

4 Viscosity Solution Approach

In this section, we shall show that the option price V defined by (10) is actually a viscosity solution of the equation (12). The general theory for viscosity solutions can be found in Crandall *et al.* [7], Fleming & Soner [8].

First, let us define the viscosity solution of (12) as follows.

Definition 4.1 *Let $w\in C([0,T]\times\mathbf{C})$. We say that w is a viscosity subsolution of (12) if, for every $\Gamma\in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathcal{D}(\mathcal{S})$, and for $(t,\psi)\in[0,T]\times\mathbf{C}$ satisfying $\Gamma\ge w$ on $[0,T]\times\mathbf{C}$ and $\Gamma(t,\psi)=w(t,\psi)$, we have*

$$\begin{aligned} &rV(t,\phi) - \frac{\partial V(t,\phi)}{\partial t} - \mathcal{S}(V)(t,\phi) - \overline{D\Gamma(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}}) \\ &-\frac{1}{2}\overline{D^2\Gamma(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}},\phi(0)g(\phi)\mathbf{1}_{\{0\}}) \le 0. \end{aligned} \tag{14}$$

We say that w is a viscosity super solution of (12) if, for every $\Gamma \in C^{1,2}_{lip}([0,T]\times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$, and for $(t,\psi) \in [0,T] \times \mathbf{C}$ satisfying $\Gamma \le w$ on $[0,T] \times \mathbf{C}$ and $\Gamma(t,\psi) = w(t,\psi)$, we have

$$
\begin{aligned}
& rV(t,\psi) - \frac{\partial V(t,\psi)}{\partial t} - \mathcal{S}(V)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\
& -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \ge 0.
\end{aligned}
\tag{15}
$$

We say that w is a viscosity solution of (12) if it is a viscosity supersolution and a viscosity subsolution of (12).

Lemma 4.2 *Let $u,t \in [0,T]$ with $t \le u$, we have*

$$
V(t,\psi) = \widetilde{\mathbf{E}}[e^{-r(u-t)}V(u,S_u)|S_t = \psi]. \tag{16}
$$

Proof Let $u,t \in [0,T]$, such that $t \le u$. Note that

$$
\begin{aligned}
V(u,S_u) &= \widetilde{\mathbf{E}}[e^{-r(T-u)}\lambda(S_T)|S_u] \\
&= \widetilde{\mathbf{E}}[e^{-r(T-u)}\lambda(S_T)|\mathcal{F}(u)].
\end{aligned}
\tag{17}
$$

In view of this, and using the fact the $S_t(\psi)$ is Markovian, we have

$$
\begin{aligned}
& \widetilde{\mathbf{E}}[e^{-r(u-t)}V(u,S_u)|S_t = \psi] \\
= {} & \widetilde{\mathbf{E}}[e^{-r(u-t)}\widetilde{\mathbf{E}}[e^{-r(T-u)}\lambda(S_T)|\mathcal{F}(u)]|S_t = \psi] \\
= {} & \widetilde{\mathbf{E}}[e^{-r(u-t)}e^{-r(T-u)}\widetilde{\mathbf{E}}[\lambda(S_T)|\mathcal{F}(u)]|\mathcal{F}(t)] \\
= {} & \widetilde{\mathbf{E}}[e^{-r(T-t)}\lambda(S_T)|\mathcal{F}(t)] \\
= {} & \widetilde{\mathbf{E}}[e^{-r(T-t)}\lambda(S_T)|S_t = \psi] \\
= {} & V(t,\psi).
\end{aligned}
\tag{18}
$$

This proves the lemma. □

Theorem 4.3 *The value function V is a viscosity solution of the infinite dimensional Black-Scholes equation*

$$
\begin{aligned}
& rV(t,\psi) - \frac{\partial V(t,\psi)}{\partial t} - \mathcal{S}(V)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\
& -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) = 0,
\end{aligned}
\tag{19}
$$

on $[0,T] \times \mathbf{C}$, and $V(T,\psi) = \Psi(\psi)$, $\forall \psi \in \mathbf{C}$.

Proof. Let $\Gamma \in C^{1,2}_{lip}([0,T]\times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$, for $(t,\psi) \in [0,T]\times \mathbf{C}$ such that $\Gamma \leq V$ on $[0,T]\times \mathbf{C}$ and $\Gamma(t,\psi) = V(t,\psi)$. We want to prove the viscosity supersolution inequality, *i.e.*,

$$\begin{aligned} & rV(t,\psi) - \frac{\partial V(t,\psi)}{\partial t} - \mathcal{S}(V)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \geq 0. \end{aligned} \tag{20}$$

Notation: throughout the proof of the theorem we will use the following notation

$$\widetilde{\mathbf{E}}^{\psi}[Y] = \widetilde{\mathbf{E}}[Y|S_t = \psi] \qquad \textit{for any process } Y.$$

Let $\Gamma \in C^{1,2}_{lip}([0,T]\times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$. For $0 \leq t \leq t_1 \leq T$, by virtue of Theorem 3.1 pp. 78 of Mohammed [19], we have

$$\begin{aligned} & \widetilde{\mathbf{E}}^{\psi}[e^{-r(t_1-t)}\Gamma(t_1, S_{t_1})] - \Gamma(t,\psi) \\ = \; & \widetilde{\mathbf{E}}^{\psi}\Bigg[\int_t^{t_1} e^{-r(u-t)}\Bigg(\frac{\partial \Gamma(u,S_u)}{\partial t} + \mathcal{S}(\Gamma)(u,S_u) + \overline{D\Gamma(u,S_u)}(rS_u(0)\mathbf{1}_{\{0\}}) \\ & +\frac{1}{2}\overline{D^2\Gamma(u,S_u)}(S_u(0)g(S_u)\mathbf{1}_{\{0\}}, S_u(0)g(S_u)\mathbf{1}_{\{0\}}) - r\Gamma(u,S_u)\Bigg)du\Bigg]. \end{aligned} \tag{21}$$

For any $t_1 \in [t,T]$, Lemma 4.2 gives,

$$V(t,\psi) \geq e^{-r(t_1-t)}\widetilde{\mathbf{E}}^{\psi}[V(t_1, S_{t_1})],$$

By virtue of (21) and using the fact that $\Gamma \leq V$, we can get

$$\begin{aligned} 0 \;\geq\; & \widetilde{\mathbf{E}}^{\psi}\Big[e^{-r(t_1-t)}V(t_1,S_{t_1})\Big] - V(t,\psi) \\ \geq\; & \widetilde{\mathbf{E}}^{\psi}\Big[e^{-r(t_1-t)}\Gamma(t_1,S_{t_1})\Big] - V(t,\psi) \\ \geq\; & \widetilde{\mathbf{E}}^{\psi}\int_t^{t_1} e^{-r(u-t)}\Bigg[\Bigg(\frac{\partial \Gamma(u,S_u)}{\partial t} + \mathcal{S}(\Gamma)(u,S_u) + \overline{D\Gamma(u,S_u)}(rS_u(0)\mathbf{1}_{\{0\}}) \\ & + \frac{1}{2}\overline{D^2\Gamma(u,S_u)}(S_u(0)g(S_u)\mathbf{1}_{\{0\}}, S_u(0)g(S_u)\mathbf{1}_{\{0\}}) - r\Gamma(u,S_u)\Bigg)\Bigg]du. \end{aligned} \tag{22}$$

Dividing by $(t_1 - t)$ and letting $t_1 \downarrow t$ in previous inequality, and we obtain

$$\begin{aligned} & rV(t,\psi) - \frac{\partial \Gamma(t,\psi)}{\partial t} - \mathcal{S}(V)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & - \frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \geq 0. \end{aligned} \tag{23}$$

So we have proved the inequality (20).

Next we want to prove that V is a viscosity subsolution. Let $\Gamma \in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathcal{D}(\mathcal{S})$. For $(t,\psi)\in[0,T]\times\mathbf{C}$ satisfying $\Gamma\geq V$ on $[0,T]\times\mathbf{C}$ and $\Gamma(t,\psi)=V(t,\psi)$, we want to prove that

$$\begin{aligned} & rV(t,\psi)-\frac{\partial\Gamma(t,\psi)}{\partial t}-\mathcal{S}(V)(t,\psi)-\overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}},\psi(0)g(\psi)\mathbf{1}_{\{0\}})\leq 0. \end{aligned} \tag{24}$$

For any $t_1\in[t,T]$, Lemma 4.2 gives,

$$V(t,\psi)\leq\widetilde{\mathbf{E}}^{\psi}e^{-r(t_1-t)}[V(t_1,S_{t_1})],$$

so we can get

$$\begin{aligned} 0 &\leq \widetilde{\mathbf{E}}^{\psi}\left[e^{-r(t_1-t)}V(t_1,S_{t_1})\right]-V(t,\psi) \\ &\leq \widetilde{\mathbf{E}}^{\psi}\left[e^{-r(t_1-t)}\Gamma(t_1,S_{t_1})\right]-\Gamma(t,\psi) \\ &\leq \widetilde{\mathbf{E}}^{\psi}\int_t^{t_1}e^{-r(u-t)}\Bigg[\Bigg(\frac{\partial\Gamma(u,S_u)}{\partial t}+\mathcal{S}(\Gamma)(u,S_u)+\overline{D\Gamma(u,S_u)}(rS_u(0)\mathbf{1}_{\{0\}}) \\ &\quad+\frac{1}{2}\overline{D^2\Gamma(u,S_u)}(S_u(0)g(S_u)\mathbf{1}_{\{0\}},S_u(0)g(S_u)\mathbf{1}_{\{0\}})-r\Gamma(u,S_u)\Bigg)\Bigg]du. \end{aligned} \tag{25}$$

Dividing by (t_1-t) and letting $t_1\downarrow t$ in previous inequality, and we obtain

$$\begin{aligned} & rV(t,\psi)-\frac{\partial\Gamma(t,\psi)}{\partial t}-\mathcal{S}(V)(t,\psi)-\overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}},\psi(0)g(\psi)\mathbf{1}_{\{0\}})\leq 0. \end{aligned} \tag{26}$$

So we have proved the inequality (24). This completes the proof of the theorem. □

Theorem 4.4 (Comparison Principle) *If $V_1(t,\psi)$ and $V_2(t,\psi)$ are both continuous with respect to the argument (t,ψ) and are respectively viscosity subsolution and supersolution of (12) with at most a linear growth. Then*

$$V_1(t,\psi)\leq V_2(t,\psi)\quad \textit{for all } (t,\psi)\in[0,T]\times\mathbf{C}. \tag{27}$$

Proof. The proof follows the same argument as in Chang *et al.* [6]. □

Given the above comparison principle, it is easy to prove that equation (12) – (13) only has a unique viscosity solution. In the next section, we will study a numerical method using finite difference method.

5 A Finite Difference Scheme

In this section, we consider an explicit finite difference scheme and show that it converges to the unique viscosity solution of equation (12). We will use a method introduced by Barles and Souganidis [2]. Given $M > 0$ and for any $(t, \phi) \in [0, T) \times \mathbf{C}_+$, we consider the equation

$$\begin{aligned} rV_M(t,\phi) &= \frac{\partial V_M(t,\phi)}{\partial t} + S(V_M)(t,\phi) + \overline{DV_M(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}}) \\ &\quad + \frac{1}{2}\overline{D^2V_M(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}}, \phi(0)g(\phi)\mathbf{1}_{\{0\}}), \\ V_M(T,\phi) &= \min(\lambda(\phi), M). \end{aligned} \tag{28}$$

Note that, as $M \to \infty$, the solution V_M of (28) converges to V, the solution of (12).

Define the function $\rho : [0, T] \to \{0, 1\}$ as follows

$$\rho(t) = \begin{cases} 1, & \text{if} \quad 0 \le t < T; \\ 0, & \text{if} \quad t = T. \end{cases} \tag{29}$$

We define the following operator

$$\begin{aligned} &\mathcal{H}(t, \phi, W, \overline{DW(t,\phi)}, \overline{D^2W(t,\phi)}) \\ = \; &\rho(t)\Bigg[rW(t,\phi) - \bigg(\frac{\partial W(t,\phi)}{\partial t} + \mathcal{S}(W)(t,\phi) \\ &\qquad + \frac{1}{2}\overline{D^2W(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}}, \phi(0)g(\phi)\mathbf{1}_{\{0\}}) \\ &\qquad + \overline{DW(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}})\bigg)\Bigg] \\ &+ (1-\rho(t))\bigg(W(t,\phi) - \min(\lambda(\phi), M)\bigg). \end{aligned} \tag{30}$$

Note that (28) is equivalent to $\mathcal{H}(t, \phi, V_M, \overline{DV_M(t,\phi)}, \overline{D^2V_M(t,\phi)}) = 0$. Let ε with $0 < \varepsilon < 1$ be the step size for variable ψ and η with $0 < \eta < 1$ be the step size for t. We consider the finite difference operators Δ_ε, Δ_η and Δ^2_η defined by

$$\begin{aligned}
\Delta_\eta W(t,\psi) &= \frac{W(t+\eta,\psi)-W(t,\psi)}{\eta}, \\
\Delta_\varepsilon W(t,\psi)(\xi+v\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(\xi+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon}, \\
\Delta^2_\varepsilon W(t,\psi)(\xi+v\mathbf{1}_{\{0\}},\zeta+w\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(\xi+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2} \\
&+ \frac{W(t,\psi-\varepsilon(\zeta+w\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2},
\end{aligned}$$

where $\xi,\zeta\in\mathbf{C}$ and $v,w\in\mathbb{R}^n$. Recall that,

$$\mathcal{S}(\Phi)(\phi)=\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].$$

Therefore we define,

$$\mathcal{S}_\varepsilon(\Phi)(\phi)=\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].$$

It is clear that $\mathcal{S}_\varepsilon(\Phi)$ is an approximation of $\mathcal{S}(\Phi)$.

We have the following lemma:

Lemma 5.1 *For any $W:[0,T]\times\mathbf{C}\to\mathbb{R}$, $W\in C^{1,2}([0,T]\times\mathbf{C})$ such that W can be smoothly extended to $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$, we have*

$$\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)(\xi+v\mathbf{1}_{\{0\}})=\overline{DW(t,\psi)}(\xi+v\mathbf{1}_{\{0\}}), \tag{31}$$

and

$$\lim_{\varepsilon\to 0}\Delta^2_\varepsilon W(t,\psi)(\xi+v\mathbf{1}_{\{0\}})=\overline{D^2W(t,\psi)}(\xi+v\mathbf{1}_{\{0\}},\zeta+w\mathbf{1}_{\{0\}}). \tag{32}$$

Proof. Note that the function W can be extended from $[0,T]\times\mathbf{C}$ to $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$. Let us denote by $\widetilde{W}$ a smooth extension of W on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$. It is clear that $\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)=d_G\widetilde{W}(t,\psi)$ where $d_G\widetilde{W}$ denote the Gâteau derivative of $\widetilde{W}$ with respect to its second variable. And since $\widetilde{W}$ is smooth then the Gâteau derivative and the Fréchet derivative of $\widetilde{W}$

coincide and are continuous extension of the DW, the Fréchet derivative of W. The uniqueness of the linear continuous extension then we have result

$$\begin{aligned}\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)(\xi+v\mathbf{1}_{\{0\}}) &= \lim_{\varepsilon\to 0}\Delta_\varepsilon \widetilde{W}(t,\psi)(\xi+v\mathbf{1}_{\{0\}}) \\ &= \overline{DW(t,\psi)}(\xi+v\mathbf{1}_{\{0\}}). \end{aligned} \tag{33}$$

Similarly argument for can used for (32). □

Let $\varepsilon,\eta>0$. The corresponding discrete version of equation (28) is given by

$$\begin{aligned} rV_M(t,\psi) &= \frac{1}{\varepsilon}\Big[V_M(t,\tilde{\psi}_\varepsilon)-V_M(t,\psi)\Big]+\frac{V_M(t+\eta,\psi)-V_M(t,\psi)}{\eta} \\ &\quad+\frac{V_M(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})-V_M(t,\psi)}{\varepsilon} \\ &\quad+\frac{1}{2}\bigg(\frac{V_M(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})-V_M(t,\psi)}{\varepsilon^2} \\ &\qquad+\frac{V_M(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})-V_M(t,\psi)}{\varepsilon^2}\bigg). \end{aligned} \tag{34}$$

Rearranging terms, we obtain the discrete version of equation (30)

$$\begin{aligned} &\rho(t)\bigg[(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)V_M(t,\psi)-\bigg(\frac{1}{\varepsilon}V_M(t,\tilde{\psi}_\varepsilon)+\frac{V_M(t+\eta,\psi)}{\eta} \\ &\qquad+\frac{1}{2}\frac{V_M(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+V_M(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{e}_i\mathbf{1}_{\{0\}})}{\varepsilon^2} \\ &\qquad+\frac{V_M(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\bigg)\bigg] \\ &+(1-\rho(t))\Big(V_M(t,\psi)-\min(\lambda(\psi),M)\Big)=0. \end{aligned} \tag{35}$$

Let $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ denote the space of bounded continuous functions W from $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ to $\mathbb{R}$. Define a mapping $\mathcal{G}_M:(0,1)^2\times[0,T]\times\mathbf{C}\times\mathbb{R}\times C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\to\mathbb{R}$ as the following

$$\begin{aligned} \mathcal{G}_M(\varepsilon,\eta,t,\psi,x,W) &= \varepsilon\rho(t)\bigg[(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)x-\bigg(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon) \\ &\quad+\frac{1}{2}\frac{W(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{e}_i\mathbf{1}_{\{0\}})}{\varepsilon^2} \\ &\quad+\frac{W(t+\eta,\psi)}{\eta}+\frac{W(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\bigg)\bigg] \end{aligned}$$

$$+\,\varepsilon(1-\rho(t))\bigg(x-\min(\lambda(\psi),M)\bigg). \tag{36}$$

Moreover, note that the coefficients of W in $\mathcal{G}_M$ are negative. This implies that $\mathcal{G}_M$ is monotone, i.e., for all $W_1, W_2 \in C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$, $\varepsilon, \eta \in (0,1)$, $t\in[0,T]$, $\psi\in\mathbf{C}$, and $x\in\mathbb{R}$, we have

$$\mathcal{G}_M(\varepsilon,\eta,t,\psi,x,W_1) \leq \mathcal{G}_M(\varepsilon,\eta,t,\psi,x,W_2) \text{ whenever } W_1 \geq W_2.$$

Definition 5.2 The scheme $\mathcal{G}_M$ is said to be consistent if, for every $t\in[0,T]$, $\psi\in\mathbf{C}\oplus\mathbf{B}$, and for every test function $W(\cdot,\cdot)$ defined on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ such that $W\in C_b^{1,2}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$,

$$\begin{aligned}&\mathcal{H}(t,\psi,W,\overline{DW(t,\psi)},\overline{D^2W(t,\psi)})\\ =&\lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,u\to 0}\frac{\mathcal{G}_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+u,W+u)}{\varepsilon}.\end{aligned}$$

Lemma 5.3 *The scheme $\mathcal{G}_M$ defined by (36) is consistent.*

Proof. Let $W\in C_b^{1,2}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$. We write

$$\begin{aligned}\frac{\mathcal{G}_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+u,W+u)}{\varepsilon} &= \rho(\tau)\bigg[-\bigg(\frac{1}{\varepsilon}(W(\tau,\tilde{\phi}_\varepsilon)+u)\\ &+\frac{W(\tau,\phi+\varepsilon r\phi(0)\mathbf{1}_{\{0\}}))+u}{\varepsilon}+\frac{W(\tau+\eta,\phi)+u}{\eta}\\ &+\frac{1}{2}\frac{W(\tau,\phi+\varepsilon\phi(0)g(\phi)\mathbf{1}_{\{0\}})+2u+W(\tau,\phi-\varepsilon\phi(0)g(\phi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\bigg)\\ &+(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)(W(\tau,\phi)+u)\bigg]\\ &+(1-\rho(\tau))\bigg((W(\tau,\phi)+u)-\min(\lambda(\phi),M)\bigg).\end{aligned}$$

Sending $u\to 0$, $\tau\to t$, $\phi\to\psi$, $\epsilon,\eta\to 0$, we have

$$\begin{aligned}&\lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,u\to 0}\frac{\mathcal{G}_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+u,W+u)}{\varepsilon}\\ =\;&\rho(\tau)\bigg[rW(t,\psi)-\bigg(\frac{\partial W(t,\psi)}{\partial t}+S(W)(t,\psi)+\overline{DW(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}})\\ &+\frac{1}{2}\overline{D^2W(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}},\psi(0)g(\psi)\mathbf{1}_{\{0\}}\bigg)\bigg]\\ &+(1-\rho(t))\bigg(W(t,\psi)-\min(\lambda(\psi),M)\bigg)\\ =\;&\mathcal{H}(t,\psi,W,\overline{DW(t,\psi)},\overline{D^2W(t,\psi)}).\end{aligned}$$

This completes the proof. □

Note that, $\mathcal{G}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W)=0$ is equivalent to the equation

$$W(t,\psi)=\frac{1}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t))}\Bigg[\rho(t)\bigg(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)+$$

$$\frac{W(t+\eta,\psi)}{\eta}+\frac{1}{2}\frac{W(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{e}_i\mathbf{1}_{\{0\}})}{\varepsilon^2}$$

$$+\frac{W(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\bigg)+(1-\rho(t))\bigg(\min(\lambda(\psi),M)\bigg)\Bigg]. \qquad (37)$$

We define an operator $\mathcal{T}_{\varepsilon,\eta}$ on $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ as follows,

$$\mathcal{T}_{\varepsilon,\eta}W(t,\psi)\equiv\frac{1}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t))}\Bigg[\rho(t)\bigg(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)$$

$$+\frac{W(t+\eta,\psi)}{\eta}+\frac{1}{2}\frac{W(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{e}_i\mathbf{1}_{\{0\}})}{\varepsilon^2}$$

$$+\frac{W(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\bigg)+(1-\rho(t))\bigg(\min(\lambda(\psi),M)\bigg)\Bigg]. \qquad (38)$$

Lemma 5.4 *For each ε and η, $\mathcal{T}_{\varepsilon,\eta}$ is a contraction on $C_b([0,T)\times(\mathbf{C}\oplus\mathbf{B}))$. map.*

Proof. To prove that $\mathcal{T}_{\epsilon,\eta}$ is a contraction, we need to show that there exists $0<\beta<1$ such that

$$\|\mathcal{T}_{\varepsilon,\eta}W_1-\mathcal{T}_{\varepsilon,\eta}W_2\|\le\beta\|W_1-W_2\| \text{ for all } W_1,W_2\in C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B})),$$

where $\|\cdot\|$ is the sup norm define by (4). Let us define $c_{\varepsilon,\eta}:[0,T]\to\mathbb{R}$ by

$$c_{\varepsilon,\eta}(t)=\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t)).$$

Note that

$$\begin{aligned}
&|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi)-\mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)|\\
&\le\Bigg[\frac{1}{c_{\varepsilon,\eta}(t)}\bigg|\rho(t)\bigg(\frac{1}{\varepsilon}W_1(t,\tilde{\psi}_\varepsilon)+\frac{W_1(t,\psi+\varepsilon r\psi(0)\mathbf{1}_{\{0\}}))}{\varepsilon}+\frac{W_1(t+\eta,\psi)}{\eta}\\
&\quad+\frac{1}{2}\frac{W_1(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W_1(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\bigg)\\
&\quad-\rho(t)\bigg(\frac{1}{\varepsilon}W_2(t,\tilde{\psi}_\varepsilon)+\frac{W_2(t,\psi+\varepsilon r\psi(0)\mathbf{1}_{\{0\}}))}{\varepsilon}+\frac{W_2(t+\eta,\psi)}{\eta}\\
&\quad+\frac{1}{2}\frac{W_2(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W_2(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\bigg)\bigg|\Bigg].
\end{aligned}$$

This implies that for all t, ψ,

$$\left|\mathcal{T}_{\varepsilon,\eta} W_1(t,\psi) - \mathcal{T}_{\varepsilon,\eta} W_2(t,\psi)\right| \leq \left[\frac{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2})}{c_{\varepsilon,\eta}(t)}\right] \|W_1 - W_2\|.$$

In addition, note that, for all $t \in [0,T)$,

$$\begin{aligned} \frac{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2})}{c_{\varepsilon,\eta}(t)} &= \frac{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2})}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t))} \\ &= \frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r} < 1. \end{aligned}$$

Define

$$\beta_{\varepsilon,\eta} \equiv \frac{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}}{\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r},$$

then we have

$$\|\mathcal{T}_{\varepsilon,\eta} W_1 - \mathcal{T}_{\varepsilon,\eta} W_2\| \leq \beta_{\varepsilon,\eta} \|W_1 - W_2\|.$$

□

Definition 5.5 The scheme $\mathcal{G}_M$ is said to be **stable** if for every $\varepsilon, \eta \in (0,1)$, there exists a bounded solution $W_{\varepsilon,\eta}$ to the equation

$$\mathcal{G}_M(\varepsilon,\eta,t,\psi,W(t,\psi),W) = 0, \quad (39)$$

with the bound independent of ε, and η.

Remark 5.6 By the Banach fixed point theorem, the strict contraction $\mathcal{T}_{\varepsilon,\eta}$ has a unique fixed point that we denote by $\widetilde{W}^M_{\varepsilon,\eta} \in C_b([0,T) \times (\mathbf{C} \oplus \mathbf{B}))$. We extend $\widetilde{W}^M_{\varepsilon,\eta}$ on $[0,T] \times (\mathbf{C} \oplus \mathbf{B})$ and denote $W^M_{\varepsilon,\eta}$ its extension. For all $\psi \in \mathbf{C} \oplus \mathbf{B}$, we have

$$W^M_{\varepsilon,\eta}(t,\psi) = \begin{cases} \widetilde{W}^M_{\varepsilon,\eta}(t,\psi), & \text{if} \quad t \in [0,T), \\ \min(\lambda(\psi), M), & \text{if} \quad t = T. \end{cases}$$

Moreover the function $W^M_{\varepsilon,\eta}$ satisfies equation (37), thus it solves equation (39). Given any function $\widetilde{W}_0 \in C_b([0,T) \times (\mathbf{C} \oplus \mathbf{B}))$ with $\|\widetilde{W}_0\| < M$, we construct a sequence as follows, $\widetilde{W}_{n+1} = \mathcal{T}_{\varepsilon,\eta} \widetilde{W}_n$ for $n \geq 0$. It is clear that

$$\lim_{n \to \infty} \widetilde{W}_n = \widetilde{W}^M_{\varepsilon,\eta}.$$

By induction we have

$$\begin{aligned}
& \widetilde{W}_{n+1}(t,\psi) \\
= \;& \frac{1}{c_{\varepsilon,\eta}(t)}\Bigg[\rho(t)\Bigg(\frac{1}{\varepsilon}\widetilde{W}_n(t,\tilde{\psi}_\varepsilon)+\frac{\widetilde{W}_n(t+\eta,\psi)}{\eta}+\frac{\widetilde{W}_n(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\Bigg) \\
& +\frac{1}{2}\frac{\widetilde{W}_n(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+\widetilde{W}_n(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2} \\
& +(1-\rho(t))\Bigg(\min(\lambda(\psi),M)\Bigg)\Bigg] \\
\leq \;& \frac{\rho(t)M(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2})+(1-\rho(t))\min(\lambda(\psi),M)}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t))} \\
\leq \;& M. \qquad (40)
\end{aligned}$$

Therefore $\|\widetilde{W}^M_{\varepsilon,\eta}\|\leq M$, thus $\|W^M_{\varepsilon,\eta}\|\leq M$. This implies that our scheme $\mathcal{G}_M$ is stable.

Theorem 5.7 *Let $W^M_{\varepsilon,\eta}$ denote the solution to* (39)*. Then, as $(\varepsilon,\eta)\to 0$, the sequence $W^M_{\varepsilon,\eta}$ converges uniformly on $[0,T]\times\mathbf{C}$ to the unique viscosity solution V_M of* (28)*.*

Proof. Define

$$\begin{aligned}
W^*_M(t,\psi) &= \limsup_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W^M_{\varepsilon,\eta}(\tau,\phi) \\
W_{*M}(t,\psi) &= \liminf_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W^M_{\varepsilon,\eta}(\tau,\phi).
\end{aligned} \qquad (41)$$

We claim that W^*_M and W_{*M} are sub- and supersolutions of (28), respectively. To prove this claim, we only consider the case for W^*_M. The argument for that of W_{*M} is similar. We want to show:

$$\mathcal{H}(t,\psi,W^*_M,\overline{D\Gamma(t,\psi)},\overline{D^2\Gamma(t,\psi)})\leq 0,$$

for any test function $\Gamma\in C^{1,2}_{lip}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$ such that (t,ψ) is a strictly local maximum of $W^*_M(\tau,\phi)-\Gamma(\tau,\phi)$. Without loss of generality, we may assume that $W^*_M\leq\Gamma$ and $W^*_M(t,\psi)=\Gamma(t,\psi)$ and because of the stability of our scheme we can also assume that $\Gamma\geq 2\sup_{\varepsilon,\eta}\|W^M_{\varepsilon,\eta}\|$ outside of the ball $B((t,\psi),l)$ centered at (t,ψ) and with the radius l, where $l>0$ is such that

$$W^*_M(\tau,\phi)-\Gamma(\tau,\phi)\leq 0=W^*_M(t,\psi)-\Phi(t,\psi) \text{ for } (\tau,\phi)\in B((t,\psi),l).$$

This implies that there exist sequences $\varepsilon_n > 0$, $\eta_n > 0$, and $(\tau_n, \phi_n) \in [0,T] \times (\mathbf{C} \oplus \mathbf{B})$ such that as $n \to \infty$ we have

$$\begin{aligned} &\varepsilon_n \to 0,\ \ \eta_n \to 0,\ \ \tau_n \to t,\ \ \phi_n \to \psi,\ \ W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n) \to W^*_M(t,\psi), \\ &\text{and}\ \ (\tau_n,\phi_n)\ \ \text{is a global maximum}\ W^M_{\varepsilon_n,\eta_n} - \Gamma. \end{aligned} \tag{42}$$

Denote $\alpha_n = W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n) - \Gamma(\tau_n,\phi_n)$. Obviously $\alpha_n \to 0$ and

$$W^M_{\varepsilon_n,\eta_n}(\tau,\phi) \le \Gamma(\tau,\phi) + \alpha_n \ \ \text{for all}\ \ (\tau,\phi) \in [0,T] \times (\mathbf{C} \oplus \mathbf{B}). \tag{43}$$

We know that

$$\mathcal{G}_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n}) = 0.$$

The monotonicity of $\mathcal{G}_M$ and (43) implies

$$\begin{aligned} &\ \ \ \ \mathcal{G}_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, \Gamma(\tau_n,\phi_n) + \alpha_n, \Gamma + \alpha_n) \\ &\le \ \mathcal{G}_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n}) \\ &= \ 0. \end{aligned} \tag{44}$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathcal{G}_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n})}{\varepsilon_n} \le 0,$$

so

$$\begin{aligned} &\mathcal{H}(t,\psi,W^*_M, \overline{D\Gamma(t,\psi)}, \overline{D^2\Gamma(t,\psi)}) \\ &= \lim_{n\to\infty} \frac{\mathcal{G}_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n,\eta_n}(\tau_n,\phi_n), W^M_{\varepsilon_n,\eta_n})}{\varepsilon_n} \le 0. \end{aligned} \tag{45}$$

This proves that W^*_M is a viscosity subsolution of (28) and, similarly we can prove that W_{*M} is a viscosity supersolution. By virtue of Theorem 4.4, we can get that

$$W_{*M}(t,\psi) \ge W^*_M(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}. \tag{46}$$

On the other hand, by the definition of W_{*M}, W^*_M, it is easy to see that

$$W_{*M}(t,\psi) \le W^*_M(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Combining with (46), the above implies

$$W_{*M}(t,\psi) = W^*_M(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Since W_{*M} is a viscosity supersolution and W^*_M is a viscosity subsolution, they are also viscosity solutions of (28). Now, using the uniqueness of the viscosity solution (28), we see that $V_M = W^*_M = W_{*M}$. Therefore, we conclude that the sequence $(W^M_{\varepsilon,\eta})_{\varepsilon,\eta}$ converges uniformly to V_M as desired. □

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