An Approximation Scheme for Black-Scholes Equations with Delays*

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Abstract

This paper addresses a finite difference approximation for an infinite dimensional Black-Scholes equation obtained in Chang and Youree [5]. The equation arises from a consideration of an European option pricing problem in a market in which stock prices and the riskless asset prices have hereditary structures. Under a general condition on the payoff function of the option, it is shown that the pricing function is the unique viscosity solution of the infinite dimensional Black-Scholes equation. In addition, a finite difference approximation of the viscosity solution is provided and the convergence results are proved.

Keywords: Finite difference, Black-Scholes equation, stochastic functional differential equations, viscosity solutions.

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1 Introduction

A European contingent claim (or option) is a contract giving the buyer of the contract (or simply the buyer) the right to buy from or sell to the contract writer (or simply writer) a share of a particular stock at a pre-determined price $K > 0$ (called the strike price) at a pre-determined time $T > 0$ (called the expiration date) in the future. The right to buy (respectively, to sell) a share of the stock is called a *call* (respectively, a *put*) option. The European (call or put) option pricing problem is, briefly, to determine the fee (called the *rational price*) that the writer should receive from the buyer for the rights of the contract and also to determine the trading strategy the writer should use to hedge his positions such that the writer will be able to cover the option if it is exercised. The fee should be large enough that the writer can, with riskless investing, cover the option, but be small enough that the writer does not make an unfair (*i.e.,* riskless) profit.

The pricing of (European) options in the continuous-time market model has been a subject of extensive research in recent years. Explicit results for the European call option had been obtained (see *e.g.,* Black & Scholes [3], Harrison & Kreps [12], Harrison & Pliska [13], Merton [20, 21], Shiryaev et al [24]) for the idealized Black-Scholes (B,S)-market consists of a bond account $\{B(t), t \geq 0\}$ and a stock account $\{S(t), t \geq 0\}$ that satisfy the following two equatons, respectively:

$$
\begin{aligned}
dB(t) &= rB(t)dt, && (1)\\
dS(t) &= S(t)[\mu dt + \sigma dZ(t)], && (2)
\end{aligned}
$$

where $\{Z(t), t \geq 0\}$ is a one-dimensional standard Brownian motion, and r, μ, σ are positive constants that represent the risk-free interest rate, the average stock appreciate rate, and the stock price volatility.

In the above model, the stock price is modeled by a geometric Brownian motion, which implies the stock return has a constant drift and volatility. In addition, the stock price process is a Markov process, which assumes independence of the stock price history. However, in the real world, the market implied volatility is not a constant, and a famous 'smile' phenomenon has been observed for European options. On the other hand, because investors tends to look at the historic performance of a stock before they decide to invest on the stock, the assumption of Markovian stock price is not very realistic.

In order to better model real-life options, in this paper, we consider a stock price model where the drift and volatility both depends on the stock price history. This model has been considered in Chang & Youree [4], [5], and similar market models with hereditary structures or delayed responses have been considered by other researchers, e.g. Arriojas et al [1], Kazmerchuk et al [15, 16, 17], in which various versions of stochastic functional differential equations were introduced to describe the dynamics of riskless assets and stock prices.

This paper addresses the issues of the viscosity solution and its finite difference approximation for an infinite dimensional Black-Scholes equation obtained from Chang and Youree [5]. The infinite dimensional Black-Scholes equation arises from the hereditary market model in which the stock price process $\{S(t), t \geq -h\}$ satisfies the following nonlinear stochastic functional differential equation

$$\frac{dS(t)}{S(t)} = f(S_t)dt + g(S_t)dZ(t), \ t \geq 0, \tag{3}$$

where $Z(t)$ is a 1-dimensional standard Brownian motion.

In addition to the nonlinearity of equation (3), one distinct feature is that the stock appreciation $f(S_t)$ and stock volatility $g(S_t)$ at time $t \geq 0$ are given nonlinear functions of the stock prices S_t (see Definition 2.1) over the time interval $[t-h, t]$ instead of just the stock price $S(t)$ at time t. The distinction between $S(t)$ and S_t shall be made clear in Section 2.

It is clear that when $h = 0$, equation (3) reduces to the following nonlinear stochastic ordinary differential equation

$$\frac{dS(t)}{S(t)} = f(S(t))dt + g(S(t))dZ(t), \ t \geq 0, \tag{4}$$

of which the geometric Brownian motion is a special case. When $h > 0$, equation (3) is general enough to include the linear model considered by Chang & Youree [4], and pure discrete delay models considered by Arriojas et al [1], and Kazmerchuk et al [15, 16, 17].

Under the hereditary market model described by equation (3), with a general payoff function $\lambda(S_T)$ at the expiration time T, Chang and Youree [5] derived an infinite-dimensional Black-Scholes equation (see equations (35) and (36) in that paper) for the pricing function for the European options. This generalized Black-Scholes equation involves extended Fréchet derivatives of functions on the function space $\mathbf{C}$ (see Section 3.1 for definitions). Under certain sufficiently smooth conditions, it was shown that the pricing function is a classical solution of the generalized Black-Scholes equation. In there, an algorithm for computing the solution of the infinite-dimensional Black-Scholes equation is also obtained via a double sequence of polynomials of a certain bounded linear functional on a Banach space and the time variable.. This paper shows, under a general payoff function, that the pricing function is the unique viscosity solution of the infinite dimensional Black-Scholes equation obtained in Chang and Youree [5]. In addition, a finite difference approximation result based on the result in Barles and Souganidis [2] is obtained in this paper.

This paper is organized as follows. Section 2 summarizes the definitions and key ingredients that will be used throughout this paper. The concepts of Fréchet derivative and extended Fréchet derivative are introduced in Section 3, along with results needed to make use of these derivatives. This section also state the infinite-dimensional Black-Scholes equation obtained in Chang and Youree [5] under the differentiability assumption of the pricing function. Section 4 shows that the pricing function is the unique viscosity solution of the infinite dimensional Black-Scholes equation. A finite difference approximation result in obtained in Section 5. An algorithm using the scheme is given in Section 6.

2 Problem Formulation

Let $h > 0$ be a fixed constant, and let $J = [-h, 0]$ denote the duration of the bounded memory of the stochastic functional differential equations considered in this paper. For the sake of simplicity, denote $C(J; \mathbb{R})$, the space of continuous functions $\phi : J \to \mathbb{R}$, by $\mathbf{C}$. Note that $\mathbf{C}$ is a real separable Banach space under the sup-norm defined by

$$\|\phi\| = \sup_{t \in J} |\phi(t)|, \quad \forall \phi \in \mathbf{C}, \tag{5}$$

where $|\cdot|$ is the Euclidean norm (absolute value) in $\mathbb{R}$. In addition, we define

$$\mathbf{C}_+ \equiv \{\phi \in \mathbf{C} | \phi(\theta) \geq 0, \forall \theta \in [-h, 0]\} \tag{6}$$

We denote by $(\cdot | \cdot)$ the inner product in $L^2(J, \mathbb{R})$ as the following

$$(\phi|\psi) = \int_{-\delta}^{0} \phi(s)\psi(s) ds, \qquad \forall \phi, \psi \in \mathbf{C}.$$

In addition, we define

$$\|\phi\|_2 = (\phi|\phi)^{\frac{1}{2}},$$

as the norm of ϕ in $L^2(J, \mathbb{R})$. Note that the space $\mathbf{C}$ can be continuously embedded into $L^2(J; \mathbb{R})$.

Let $L^2(\Omega, \mathbf{C})$ be the space of $\mathbf{C}$-valued random variables $\Theta : \Omega \to \mathbf{C}$ such that

$$\|\Theta\|_{L^2} = \left\{ \int_{\Omega} \|\Theta(\omega)\|^2 dP(\omega) \right\}^{\frac{1}{2}} < \infty.$$

Let $L^2(\Omega, \mathbf{C}; \mathcal{F}(t))$ be those $\Theta \in L^2(\Omega, \mathbf{C})$ which are $\mathcal{F}(t)$-measurable.

Definition 2.1 *If $\psi \in C([-h, \infty); \mathbb{R})$ and $t \in \mathbb{R}_+$, let $\psi_t \in \mathbf{C}$ be defined by (see Hale [11]):*

$$\psi_t(\theta) \equiv \psi(t + \theta), \quad \forall \theta \in [-h, 0].$$

Let $\{Z(t), t \geq 0\}$ be a 1-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}(t), t \geq 0\}$ is the P-augmentation of the natural filtration $\{\mathcal{F}^Z(t), t \geq 0\}$ generated by the Brownian motion $\{Z(t), t \geq 0\}$, *i.e.*, if $t \geq 0$,

$$\mathcal{F}^Z(t) = \sigma\{Z(s), 0 \leq s \leq t\}$$

and

$$\mathcal{F}(t) = \mathcal{F}^Z(t) \vee \{A \subset \Omega | A \subset B, B \in \mathcal{F}, P(B) = 0\}$$

where the operator $\vee$ denotes that $\mathcal{F}(t)$ is the smallest σ-algebra such that $\mathcal{F}^Z(t) \subset \mathcal{F}(t)$ and $\{A \subset \Omega | A \subset B, B \in \mathcal{F}, P(B) = 0\} \subset \mathcal{F}(t)$.

Let $T > 0$ be the expiration time of the European option. We assume that the stock price process $\{S(t), t \in [-h, T]\}$ satisfies the following nonlinear stochastic functional differential equation:

$$\frac{dS(t)}{S(t)} = f(S_t)dt + g(S_t)dZ(t), \quad \forall t \in (0, T] \tag{7}$$

and the initial condition

$$S_0 = \psi, \tag{8}$$

which is equivalent to

$$S(\theta) = \psi(\theta), \quad \forall \theta \in [-h, 0],$$

and $\psi \in \mathbf{C}_+$ is the initial price function. The functions, $f : \mathbf{C} \to \mathbb{R}$ and $g : \mathbf{C} \to \mathbb{R}$ are given deterministic functions.

Definition 2.2 *A process $\{S(t; \psi), t \in [-h, T]\}$ is said to be a (strong) solution of $(7)-(8)$ on the interval $[-h, T]$ and through the initial price function $\psi \in \mathbf{C}$ if it satisfies the following conditions:*

1. $S(t) = \psi(t)$ *for all $t \in [-h, 0]$;*
2. $S(t; \psi)$ *is $\mathcal{F}(t)$-measurable for each $t \in [0, T]$;*
3. *The process $\{S(t; \psi), t \in [0, T]\}$ is continuous and satisfies the following stochastic integral equation P-a.s.*

$$S(t) = \psi(0) + \int_0^t S(u) f(S_u) du + \int_0^t S(u) g(S_u) dZ(u), \ T \geq t \geq 0. \tag{9}$$

In addition, the solution process $\{S(t;\psi), t \in [-h,T]\}$ is said to be (strongly) unique if $\{\tilde{S}(t;\psi), t \in [-h,T]\}$ is also a solution of (7) on $[-h,T]$ and through the same initial datum ψ, then

$$P\{S(t;\psi) = \tilde{S}(t;\psi), \forall t \in [0,T]\} = 1.$$

In addition, we assume that the functions $f : \mathbf{C} \to \mathbb{R}$, and $g : \mathbf{C} \to \mathbb{R}$ satisfy the following conditions to ensure the existence and uniqueness of a global (strong) solution $\{S(t;\psi), t \in [-h,T]\}$ for each $\psi \in L^2(\Omega, \mathbf{C})$. (See Mohammed [22, 23].) The functions f and g are continuous and they satisfy the following linear growth and Lipschitz conditions.

Assumption 2.3 *There exists a constant $\Lambda > 0$ such that*

$$|\varphi(0)f(\varphi) - \phi(0)f(\phi)| + |\varphi(0)g(\varphi) - \phi(0)g(\phi)| \le \Lambda \|\varphi - \phi\|, \forall \varphi, \phi \in \mathbf{C}.$$

Assumption 2.4 *There exists a constant $K > 0$ such that*

$$|\phi(0)f(\phi)| + |\phi(0)g(\phi)| \le K(1 + \|\phi\|), \ \forall \phi \in \mathbf{C}.$$

We want to point out that the $\mathbf{C}$-valued process $\{S_t(\psi)\}_{0 \le t \le T}$ is a Markov process (see Mohammed [22], [23]).

On the other hand, we assume that the price of the riskless asset $B(t)$ has delayed effect, too. In particular, we assume that it satisfies the following equation:

$$dB(t) = L(B_t)dt, \quad t \ge 0, \tag{10}$$

with an initial function $\phi \in \mathbf{C}_+$. In the above equation, L is a bounded linear functional that can be represented as (see [11, 5]):

$$L(\phi) = \int_{-h}^{0} \phi(\theta) d\xi(\theta),$$

for some function $\xi : [-h, 0] \to \mathbb{R}$ that satisfies the following assumption:

Assumption 2.5 *The function $\xi : [-h, 0] \to \mathbb{R}$ is nondecreasing (and hence of bounded variation) with $\xi(0) - \xi(-h) > 0$.*

To describe the returns on the riskless asset described by (10), we assume that the solution $B(t), t \in [-h, \infty)$ takes the following form:

$$B(t) = \phi(0)e^{rt},$$

where $r > 0$ is the equivalent interest rate. It is easy to verify that r satisfies the satisfies the following equation:

$$r = \int_{-h}^{0} e^{r\theta} d\xi(\theta), \tag{11}$$

Under Assumption 2.5, the existence and uniqueness of a positive number r that satisfies the above equation is shown in Chang and Youree [4] (Proposition 2.5).

Now consider the non-arbitrage price of a European type option. Given a reward/payoff function $\lambda : \mathbf{C} \to \mathbb{R}_+$, the price function $V : [0,T] \times \mathbf{C} \to \mathbb{R}$ is defined as follows:

$$V(t,\psi) \quad = \quad \widetilde{\mathbf{E}}\Big[e^{-r(T-t)}\lambda(S_T)|S_t = \psi\Big] \tag{12}$$

where $\widetilde{\mathbf{E}}$ represents the expectation for a suitable (risk-neutral) probability measure $\tilde{P}$ defined in Lemma 2.8 of Chang and Youree [5]. Given a continuous payoff λ with linear growth rate, one can show that the pricing function V is also a continuous function with linear growth rate.

The reward function $\lambda : \mathbf{C} \to \mathbb{R}$ can be a very general measurable function. In particular, it can include the following special cases.

Example 2.6 *(i) $\lambda(\psi) = (\psi(0)-K)^+$. This is the reward function for a standard European call option. Note that this is also the payoff function considered in Arriojas et al [1].*
(ii) $\lambda(\psi) = (K - \psi(0))^+$. This is the payoff function for a standard European put option.
(iii) $\lambda(\psi) = \int_{-h}^{0} H(\psi(\theta)e^{-r\theta}, \psi(0))d\theta$ for some function $H : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. This is the payoff function for the European option considered in Kazmerchuk et al [15, 16, 17]. Note that the reward function $\lambda \in C^2(\mathbf{C})$ if the function $H \in C^{2,2}(\mathbb{R}\times\mathbb{R})$ (i.e., $H(x,y)$ is twice continuously differentiable with respect to x and y).

3 Infinite-Dimensional Black-Scholes Equation

3.1 The Infinitesimal Generator

Let $\mathbf{C}^*$ and $\mathbf{C}^\dagger$ be the space of bounded linear functionals $\Phi : \mathbf{C} \to \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : \mathbf{C}\times\mathbf{C} \to \mathbb{R}$, of the space $\mathbf{C}$, respectively. They are equipped with the operator norms which will be, respectively, denoted by $\|\cdot\|^*$ and $\|\cdot\|^\dagger$.

Let $\mathbf{B} = \{v\mathbf{1}_{\{0\}}, v \in \mathbb{R}\}$, where $\mathbf{1}_{\{0\}} : [-h,0] \to \mathbb{R}$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-h,0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$\mathbf{C} \oplus \mathbf{B} = \{\phi + v\mathbf{1}_{\{0\}} \mid \phi \in \mathbf{C}, v \in \mathbb{R}\}$$

and equip it with the norm $\|\cdot\|$ defined by

$$\|\phi + v\mathbf{1}_{\{0\}}\| = \sup_{\theta\in[-h,0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbf{C}, v \in \mathbb{R}.$$

Note that for each sufficiently smooth function $\Phi : \mathbf{C} \to \mathbb{R}$, its first order Fréchet derivative (with respect to $\varphi \in \mathbf{C}$), $D\Phi(\varphi) \in \mathbf{C}^*$, has a unique and continuous linear extension $\overline{D\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi) \in \mathbf{C}^\dagger$, has a unique and continuous linear extension $\overline{D^2\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^\dagger$, where $(\mathbf{C} \oplus \mathbf{B})^*$and $(\mathbf{C} \oplus \mathbf{B})^\dagger$ are spaces of bounded linear and bilinear functionals of $\mathbf{C} \oplus \mathbf{B}$, respectively. (See Lemma 3.1 and Lemma 3.2 on pp 79-83 of Mohammed [22] for details).

For a Borel measurable function $\Phi : \mathbf{C} \to \mathbb{R}$, we also define

$$\mathcal{S}(\Phi)(\phi) \equiv \lim_{u \to 0+} \frac{1}{u} \left[\Phi(\tilde{\phi}_u) - \Phi(\phi) \right] \tag{13}$$

for all $\phi \in \mathbf{C}$, where $\tilde{\phi} : [-h, T] \to \mathbb{R}$ is an extension of ϕ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-h, 0) \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and $\tilde{\phi}_t \in \mathbf{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-h, 0].$$

Let $\mathcal{D}(\mathcal{S})$, the domain of the operator $\mathcal{S}$, be the set of $\Phi : \mathbf{C} \to \mathbb{R}$ such that the above limit exists for each $\phi \in \mathbf{C}$.

Let $C^{1,2}_{lip}([0, T] \times \mathbf{C})$ be the space of functions $\Phi : [0, T] \times \mathbf{C} \to \mathbb{R}$ such that $\frac{\partial \Phi}{\partial t} : [0, T] \times \mathbf{C} \to \mathbb{R}$ and $D^2\Phi : [0, T] \times \mathbf{C} \to \mathbf{C}^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t, \phi) - D^2\Phi(t, \varphi)\|^\dagger \leq K\|\phi - \varphi\| \quad \forall t \in [0, T], \ \phi, \varphi \in \mathbf{C}.$$

By changing the probability measure to the risk-neutral probability measure, the equation (3) can be replaced by

$$dS(t) = S(t)[rdt + g(S_t)d\widetilde{Z}(t)], \tag{14}$$

where $\widetilde{Z}(t)$ is a standard Brownian motion under the new probability measure $\tilde{P}$. Therefore, in the sequel, we will restrict our attention to the above price process instead of (3). The details of the probability measure change from P to $\tilde{P}$ can be found in Chang and Youree [5]. In addition, we have the following result from Chang and Youree [5]:

Theorem 3.1 *Let $V(t, \psi) = e^{-r(T-t)}\widetilde{\mathbf{E}}[\lambda(S_T)|S_t = \psi]$, suppose that $V(t, \psi) \in C^{1,2}_{lip}([0, T] \times \mathbf{C}_+) \cap \mathcal{D}(\mathcal{S})$. Then for any $(t, \psi) \in [0, T) \times \mathbf{C}_+$, we have*

$$\begin{aligned} rV(t, \psi) &= \frac{\partial V(t, \psi)}{\partial t} + \mathcal{S}(V)(t, \psi) + \overline{DV(t, \psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ &\quad + \frac{1}{2}\overline{D^2V(t, \psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \end{aligned} \tag{15}$$

with

$$V(T,\psi) = \lambda(\psi). \tag{16}$$

Remark 3.2 *Since we will use the method presented in Barles-Souganidis [2], in the next section, we will prove that the option price can be obtained as a viscosity solution of the Black-Sholes equation (15-16). This fact will be crucial in proving the convergence of our scheme.*

4 Viscosity Solution Approach

In this section, we shall show that the option price V defined by (12) is actually a viscosity solution of the equation (15). For theories of viscosity solutions, please refer to crandall, Ishii and Lions [9] as well as Fleming and Soner [10].

First, let us define the viscosity solution of (15) as follows.

Definition 4.1 *Let $w \in C([0,T] \times \mathbf{C})$. We say that w is a viscosity subsolution of (15) if, for every $\Gamma \in C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$, and for $(t,\phi) \in [0,T] \times \mathbf{C}$ satisfying $\Gamma \geq w$ on $[0,T] \times \mathbf{C}$ and $\Gamma(t,\phi) = w(t,\phi)$, we have*

$$\begin{aligned} & r\Gamma(t,\phi) - \frac{\partial \Gamma(t,\phi)}{\partial t} - \mathcal{S}(\Gamma)(t,\phi) - \overline{D\Gamma(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2\Gamma(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}}, \phi(0)g(\phi)\mathbf{1}_{\{0\}}) \leq 0. \end{aligned} \tag{17}$$

We say that w is a viscosity supersolution of (15) if, for every $\Gamma \in C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$, and for $(t,\psi) \in [0,T] \times \mathbf{C}$ satisfying $\Gamma \leq w$ on $[0,T] \times \mathbf{C}$ and $\Gamma(t,\psi) = w(t,\psi)$, we have

$$\begin{aligned} & r\Gamma(t,\psi) - \frac{\partial \Gamma(t,\psi)}{\partial t} - \mathcal{S}(\Gamma)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \geq 0. \end{aligned} \tag{18}$$

We say that w is a viscosity solution of (15) if it is a viscosity supersolution and a viscosity subsolution of (15).

Lemma 4.2 *Let $u, t \in [0,T]$ with $t \leq u$ and let $\psi \in \mathbf{C}$. Then we have*

$$V(t,\psi) = \widetilde{\mathbf{E}}[e^{-r(u-t)}V(u,S_u)|S_t = \psi]. \tag{19}$$

Proof By the assumptions, we have that

$$\begin{aligned} V(u,S_u) &= \widetilde{\mathbf{E}}[e^{-r(T-u)}\lambda(S_T)|S_u] \\ &= \widetilde{\mathbf{E}}[e^{-r(T-u)}\lambda(S_T)|\mathcal{F}(u)]. \end{aligned} \tag{20}$$

In view of this, and using the fact the $S_t(\psi)$ is Markovian, we have

$$\begin{aligned} & \widetilde{\mathbf{E}}[e^{-r(u-t)}V(u,S_u)|S_t=\psi] \\ = & \widetilde{\mathbf{E}}[e^{-r(u-t)}\widetilde{\mathbf{E}}[e^{-r(T-u)}\lambda(S_T)|\mathcal{F}(u)]|S_t=\psi] \\ = & \widetilde{\mathbf{E}}[e^{-r(u-t)}e^{-r(T-u)}\widetilde{\mathbf{E}}[\lambda(S_T)|\mathcal{F}(u)]|\mathcal{F}(t)] \\ = & \widetilde{\mathbf{E}}[e^{-r(T-t)}\lambda(S_T)|\mathcal{F}(t)] \\ = & \widetilde{\mathbf{E}}[e^{-r(T-t)}\lambda(S_T)|S_t=\psi] \\ = & V(t,\psi). \end{aligned} \tag{21}$$

This proves the lemma. □

Theorem 4.3 *The value function V is a viscosity solution of the HJB equation* (15)

$$\begin{aligned} & rV(t,\psi) - \frac{\partial V(t,\psi)}{\partial t} - \mathcal{S}(V)(t,\psi) - \overline{DV(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2V(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}},\psi(0)g(\psi)\mathbf{1}_{\{0\}}) = 0, \end{aligned} \tag{22}$$

on $[0,T]\times\mathbf{C}$*, and* $V(T,\psi)=\lambda(\psi),\ \forall\psi\in\mathbf{C}$.

Proof. Let $\Gamma\in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathcal{D}(\mathcal{S})$, for $(t,\psi)\in[0,T]\times\mathbf{C}$ such that $\Gamma\le V$ on $[0,T]\times\mathbf{C}$ and $\Gamma(t,\psi)=V(t,\psi)$. We want to prove the viscosity supersolution inequality, *i.e.*,

$$\begin{aligned} & r\Gamma(t,\psi) - \frac{\partial\Gamma(t,\psi)}{\partial t} - \mathcal{S}(\Gamma)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\ & -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}},\psi(0)g(\psi)\mathbf{1}_{\{0\}}) \ge 0. \end{aligned} \tag{23}$$

Notation: throughout the proof of the theorem we will use the following notation

$$\widetilde{\mathbf{E}}^{\psi}[Y] = \widetilde{\mathbf{E}}[Y|S_t=\psi] \qquad \textit{for any process } Y.$$

Let $\Gamma\in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathcal{D}(\mathcal{S})$. For $0\le t\le t_1\le T$, by virtue of Theorem 3.1 pp. 78 of Mohammed [22], we have

$$\begin{aligned} & \widetilde{\mathbf{E}}^{\psi}[e^{-r(t_1-t)}\Gamma(t_1,S_{t_1})] - \Gamma(t,\psi) \\ = & \widetilde{\mathbf{E}}^{\psi}\Bigg[\int_t^{t_1} e^{-r(u-t)}\Bigg(\frac{\partial\Gamma(u,S_u)}{\partial t} + \mathcal{S}(\Gamma)(u,S_u) + \overline{D\Gamma(u,S_u)}(rS_u(0)\mathbf{1}_{\{0\}}) \\ & +\frac{1}{2}\overline{D^2\Gamma(u,S_u)}(S_u(0)g(S_u)\mathbf{1}_{\{0\}},S_u(0)g(S_u)\mathbf{1}_{\{0\}}) - r\Gamma(u,S_u)\Bigg)du\Bigg]. \end{aligned} \tag{24}$$

For any $t_1 \in [t, T]$, Lemma 4.2 gives,

$$V(t,\psi) \geq e^{-r(t_1-t)}\widetilde{\mathbf{E}}^{\psi}[V(t_1, S_{t_1})],$$

By virtue of (24) and using the fact that $\Gamma \leq V, \Gamma(t,\psi) = V(t,\psi)$, we can get

$$\begin{aligned}
0 &\geq \widetilde{\mathbf{E}}^{\psi}\Big[e^{-r(t_1-t)}V(t_1, S_{t_1})\Big] - V(t,\psi) \\
&\geq \widetilde{\mathbf{E}}^{\psi}\Big[e^{-r(t_1-t)}\Gamma(t_1, S_{t_1})\Big] - \Gamma(t,\psi) \\
&\geq \widetilde{\mathbf{E}}^{\psi}\int_t^{t_1} e^{-r(u-t)}\Bigg[\bigg(\frac{\partial \Gamma(u,S_u)}{\partial t} + \mathcal{S}(\Gamma)(u,S_u) + \overline{D\Gamma(u,S_u)}(rS_u(0)\mathbf{1}_{\{0\}}) \\
&\quad + \frac{1}{2}\overline{D^2\Gamma(u,S_u)}(S_u(0)g(S_u)\mathbf{1}_{\{0\}}, S_u(0)g(S_u)\mathbf{1}_{\{0\}}) - r\Gamma(u,S_u)\bigg)\Bigg]du. \qquad (25)
\end{aligned}$$

Dividing by $(t_1 - t)$ and letting $t_1 \downarrow t$ in previous inequality, and we obtain

$$\begin{aligned}
& r\Gamma(t,\psi) - \frac{\partial \Gamma(t,\psi)}{\partial t} - \mathcal{S}(\Gamma)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\
& - \frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \geq 0. \qquad (26)
\end{aligned}$$

So we have proved the inequality (23).

Next we want to prove that V is a viscosity subsolution. Let $\Gamma \in C^{1,2}_{lip}([0,T]\times\mathbf{C})\cap\mathcal{D}(\mathcal{S})$. For $(t,\psi) \in [0,T]\times\mathbf{C}$ satisfying $\Gamma \geq V$ on $[0,T]\times\mathbf{C}$ and $\Gamma(t,\psi) = V(t,\psi)$, we want to prove that

$$\begin{aligned}
& r\Gamma(t,\psi) - \frac{\partial \Gamma(t,\psi)}{\partial t} - \mathcal{S}(\Gamma)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\
& - \frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \leq 0. \qquad (27)
\end{aligned}$$

For any $t_1 \in [t, T]$, Lemma 4.2 gives,

$$V(t,\psi) \leq \widetilde{\mathbf{E}}^{\psi}e^{-r(t_1-t)}[V(t_1, S_{t_1})].$$

Using the fact that $\Gamma \leq V, \Gamma(t,\psi) = V(t,\psi)$, then we can get

$$\begin{aligned}
0 &= \widetilde{\mathbf{E}}^{\psi}\Big[e^{-r(t_1-t)}V(t_1, S_{t_1})\Big] - V(t,\psi) \\
&\leq \widetilde{\mathbf{E}}^{\psi}\Big[e^{-r(t_1-t)}\Gamma(t_1, S_{t_1})\Big] - \Gamma(t,\psi) \\
&\leq \widetilde{\mathbf{E}}^{\psi}\int_t^{t_1} e^{-r(u-t)}\Bigg[\bigg(\frac{\partial \Gamma(u,S_u)}{\partial t} + \mathcal{S}(\Gamma)(u,S_u) + \overline{D\Gamma(u,S_u)}(rS_u(0)\mathbf{1}_{\{0\}}) \\
&\quad + \frac{1}{2}\overline{D^2\Gamma(u,S_u)}(S_u(0)g(S_u)\mathbf{1}_{\{0\}}, S_u(0)g(S_u)\mathbf{1}_{\{0\}}) - r\Gamma(u,S_u)\bigg)\Bigg]du. \qquad (28)
\end{aligned}$$

Dividing by $(t_1 - t)$ and letting $t_1 \downarrow t$ in previous inequality, and we obtain

$$
\begin{aligned}
& r\Gamma(t,\psi) - \frac{\partial \Gamma(t,\psi)}{\partial t} - \mathcal{S}(\Gamma)(t,\psi) - \overline{D\Gamma(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}}) \\
& -\frac{1}{2}\overline{D^2\Gamma(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}}, \psi(0)g(\psi)\mathbf{1}_{\{0\}}) \leq 0. \qquad (29)
\end{aligned}
$$

So we have proved the inequality (27). This completes the proof of the theorem. □

The following comparison result is needed to prove the uniqueness result:

Theorem 4.4 (Comparison Principle) *If $V_1(t,\psi)$ and $V_2(t,\psi)$ are both continuous with respect to the argument (t,ψ) and are respectively viscosity subsolution and supersolution of (15) with at most a linear growth. Then*

$$
V_1(t,\psi) \leq V_2(t,\psi) \quad \textit{for all } (t,\psi) \in [0,T] \times \mathbf{C}. \qquad (30)
$$

Proof. The proof follows the same argument as in Chang *et al.* [8]. □

Given the above comparison principle, it is easy to prove that equation $(15-16)$ only has a unique viscosity solution. In next section, we will consider a numerical method using finite difference method.

5 A Finite Difference Scheme

In this section, we consider an explicit finite difference scheme and show that it converges to the unique viscosity solution of equation (15). We will use a method introduced by Barles and Souganidis [2]. Similar finite difference schemes have been consider in the authors' other papers [6, 7] for stochastic control problems and optimal stopping problems with delays.

Given $M > 0$ and for any $(t,\phi) \in [0,T) \times \mathbf{C}_+$, we consider the equation

$$
\begin{aligned}
rV_M(t,\phi) &= \frac{\partial V_M(t,\phi)}{\partial t} + S(V_M)(t,\phi) + \overline{DV_M(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}}) \\
&\quad + \frac{1}{2}\overline{D^2V_M(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}}, \phi(0)g(\phi)\mathbf{1}_{\{0\}}), \qquad (31) \\
V_M(T,\phi) &= \min(\lambda(\phi), M).
\end{aligned}
$$

Note that, as M goes to infinity the solution V_M of (31) converges to V the solution of (15).

Define the function $\rho : [0,T] \to \{0,1\}$ as follows

$$
\rho(t) = \begin{cases} 1, & \text{if} \quad 0 \leq t < T; \\ 0, & \text{if} \quad t = T. \end{cases} \qquad (32)
$$

We define the following operator

$$
\begin{aligned}
& \mathcal{H}(t,\phi,W,\overline{DW(t,\phi)},\overline{D^2W(t,\phi)}) \\
= \;& \rho(t)\Bigg[rW(t,\phi)-\Bigg(\frac{\partial W(t,\phi)}{\partial t}+\mathcal{S}(W)(t,\phi) \\
& \quad +\frac{1}{2}\overline{D^2W(t,\phi)}(\phi(0)g(\phi)\mathbf{1}_{\{0\}},\phi(0)g(\phi)\mathbf{1}_{\{0\}}) \\
& \quad +\overline{DW(t,\phi)}(r\phi(0)\mathbf{1}_{\{0\}})\Bigg)\Bigg] \\
& +(1-\rho(t))\Bigg(W(t,\phi)-\min(\lambda(\phi),M)\Bigg). \qquad (33)
\end{aligned}
$$

Note that (31) is equivalent to

$$
\mathcal{H}(t,\phi,V_M,\overline{DV_M(t,\phi)},\overline{D^2V_M(t,\phi)})=0.
$$

Let ε with $0<\varepsilon<1$ be the stepsize for variable ψ and η with $0<\eta<1$ be the stepsize for t. We consider the finite difference operators Δ_η, Δ_ε and Δ_ε^2 defined by

$$
\begin{aligned}
\Delta_\eta W(t,\psi) &= \frac{W(t+\eta,\psi)-W(t,\psi)}{\eta}, \\
\Delta_\varepsilon W(t,\psi)(l+v\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(l+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon}, \\
\Delta_\varepsilon^2 W(t,\psi)(l+v\mathbf{1}_{\{0\}},k+w\mathbf{1}_{\{0\}}) &= \frac{W(t,\psi+\varepsilon(l+v\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2} \\
&+ \frac{W(t,\psi-\varepsilon(k+w\mathbf{1}_{\{0\}}))-W(t,\psi)}{\varepsilon^2},
\end{aligned}
$$

where $l,k\in\mathbf{C}$ and $v,w\in\mathbb{R}$. Recall that,

$$
\mathcal{S}(\Phi)(\phi)=\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].
$$

Therefore we can define,

$$
\mathcal{S}_\varepsilon(\Phi)(\phi)=\frac{1}{\varepsilon}\left[\Phi(\tilde{\phi}_\varepsilon)-\Phi(\phi)\right].
$$

It is clear that $\mathcal{S}_\varepsilon(\Phi)$ is an approximation of $\mathcal{S}(\Phi)$.

We have the following lemma:

Lemma 5.1 *For any $W:[0,T]\times\mathbf{C}\to\mathbb{R}$, $W\in C^{1,2}([0,T]\times\mathbf{C})$ such that W can be smoothly extended on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$, we have*

$$
\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)(l+v\mathbf{1}_{\{0\}})=\overline{DW(t,\psi)}(l+v\mathbf{1}_{\{0\}}), \qquad (34)
$$

and

$$\lim_{\varepsilon\to 0}\Delta_\varepsilon^2 W(t,\psi)(l+v\mathbf{1}_{\{0\}}) = \overline{D^2W(t,\psi)}(l+v\mathbf{1}_{\{0\}},k+w\mathbf{1}_{\{0\}}). \tag{35}$$

Proof. Note that the function W can be extended from $[0,T]\times\mathbf{C}$ to $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$. Let us denote by $\widetilde{W}$ a smooth extension of W on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$. It is clear that $\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi) = d_G\widetilde{W}(t,\psi)$ where $d_G\widetilde{W}$ denote the Gâteau derivative of $\widetilde{W}$ with respect to its second variable. In addition, because $\widetilde{W}$ is smooth, the Gâteau derivative and the Fréchet derivative of $\widetilde{W}$ coincide and are continuous extension of the DW, the Fréchet derivative of W. The uniqueness of the linear continuous extension then we have result

$$\begin{aligned}\lim_{\varepsilon\to 0}\Delta_\varepsilon W(t,\psi)(l+v\mathbf{1}_{\{0\}}) &= \lim_{\varepsilon\to 0}\Delta_\varepsilon\widetilde{W}(t,\psi)(l+v\mathbf{1}_{\{0\}})\\ &= \overline{DW(t,\psi)}(l+v\mathbf{1}_{\{0\}}).\end{aligned} \tag{36}$$

Similarly argument for can used to prove (35). □

Let $\varepsilon,\eta>0$. The corresponding discrete version of equation (31) is given by

$$\begin{aligned}rV_M(t,\psi) &= \frac{1}{\varepsilon}\Big[V_M(t,\tilde{\psi}_\varepsilon)-V_M(t,\psi)\Big]+\frac{V_M(t+\eta,\psi)-V_M(t,\psi)}{\eta}\\ &+\frac{V_M(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})-V_M(t,\psi)}{\varepsilon}\\ &+\frac{1}{2}\bigg(\frac{V_M(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})-V_M(t,\psi)}{\varepsilon^2}\\ &\qquad+\frac{V_M(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})-V_M(t,\psi)}{\varepsilon^2}\bigg).\end{aligned} \tag{37}$$

Rearranging terms and using the terminal condition of (31), we obtain the discrete version of equation (33)

$$\begin{aligned}&\rho(t)\bigg[(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)V_M(t,\psi)-\bigg(\frac{1}{\varepsilon}V_M(t,\tilde{\psi}_\varepsilon)+\frac{V_M(t+\eta,\psi)}{\eta}\\ &\qquad+\frac{1}{2}\frac{V_M(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+V_M(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\\ &\qquad+\frac{V_M(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\bigg)\bigg]\\ &+(1-\rho(t))\bigg(V_M(t,\psi)-\min(\lambda(\psi),M)\bigg)=0.\end{aligned} \tag{38}$$

Let $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ denote the space of bounded continuous functions W from $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ to $\mathbb{R}$. Define a mapping $S_M:(0,1)^2\times[0,T]\times\mathbf{C}\times\mathbb{R}\times C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\to\mathbb{R}$ as the following

$$\begin{aligned}
S_M(\varepsilon,\eta,t,\psi,x,W) &= \varepsilon\rho(t)\bigg[\left(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r\right)x-\bigg(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)\\
&+\frac{1}{2}\frac{W(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\\
&+\frac{W(t+\eta,\psi)}{\eta}+\frac{W(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\bigg)\bigg]\\
&+\varepsilon(1-\rho(t))\Big(x-\min(\lambda(\psi),M)\Big).
\end{aligned}\tag{39}$$

Moreover, note that the coefficients of W in S_M are negative. This implies that S_M is monotone in W, i.e., for all $W_1,W_2\in C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$, $\varepsilon,\eta\in(0,1)$, $t\in[0,T]$, $\psi\in\mathbf{C}$, and $x\in\mathbb{R}$, we have

$$S_M(\varepsilon,\eta,t,\psi,x,W_1)\le S_M(\varepsilon,\eta,t,\psi,x,W_2) \text{ whenever } W_1\ge W_2.$$

Definition 5.2 The scheme S_M is said to be consistent if, for every $t\in[0,T]$, $\psi\in\mathbf{C}\oplus\mathbf{B}$, and for every test function $W(\cdot,\cdot)$ defined on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ such that $W\in C_b^{1,2}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$,

$$\begin{aligned}
&\mathcal{H}(t,\psi,W,\overline{DW(t,\psi)},\overline{D^2W(t,\psi)})\\
&=\lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,\xi\to 0}\frac{S_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon},
\end{aligned}$$

where $\mathcal{H}$ is defined by (33).

We have the following lemma:

Lemma 5.3 *The scheme S_M defined by (39) is consistent.*

Proof. Let $W\in C_b^{1,2}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$. We write

$$\begin{aligned}
\frac{S_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon} &= \rho(\tau)\bigg[-\bigg(\frac{1}{\varepsilon}(W(\tau,\tilde{\phi}_\varepsilon)+\xi)\\
&+\frac{W(\tau,\phi+\varepsilon r\phi(0)\mathbf{1}_{\{0\}})+\xi}{\varepsilon}+\frac{W(\tau+\eta,\phi)+\xi}{\eta}\\
&+\frac{1}{2}\frac{W(\tau,\phi+\varepsilon\phi(0)g(\phi)\mathbf{1}_{\{0\}})+2\xi+W(\tau,\phi-\varepsilon\phi(0)g(\phi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\bigg)
\end{aligned}$$

$$+(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)(W(\tau,\phi)+\xi)\Big]$$
$$+(1-\rho(\tau))\Big((W(\tau,\phi)+\xi)-\min(\lambda(\phi),M)\Big).$$

Sending $\xi\to 0$, $\tau\to t$, $\phi\to\psi$, $\varepsilon,\eta\to 0$, we have

$$\begin{aligned}
&\lim_{(\tau,\phi)\to(t,\psi),\,\varepsilon,\eta\downarrow 0,\,\xi\to 0}\frac{S_M(\varepsilon,\eta,\tau,\phi,W(\tau,\phi)+\xi,W+\xi)}{\varepsilon}\\
&=\rho(\tau)\Big[rW(t,\psi)-\Big(\frac{\partial W(t,\psi)}{\partial t}+\mathcal{S}(W)(t,\psi)+\overline{DW(t,\psi)}(r\psi(0)\mathbf{1}_{\{0\}})\\
&\quad+\frac{1}{2}\overline{D^2W(t,\psi)}(\psi(0)g(\psi)\mathbf{1}_{\{0\}},\psi(0)g(\psi)\mathbf{1}_{\{0\}}\Big)\Big]\\
&\quad+(1-\rho(t))\Big(W(t,\psi)-\min(\lambda(\psi),M)\Big)\\
&=\mathcal{H}(t,\psi,W,\overline{DW(t,\psi)},\overline{D^2W(t,\psi)}).
\end{aligned}$$

This completes the proof. □

Note that, $S_M(\varepsilon,\eta,t,\psi,W(t,\psi),W)=0$ is equivalent to the equation

$$\begin{aligned}
W(t,\psi)&=\frac{1}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t))}\Big[\rho(t)\Big(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)+\\
&\frac{W(t+\eta,\psi)}{\eta}+\frac{1}{2}\frac{W(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\\
&+\frac{W(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\Big)+(1-\rho(t))\Big(\min(\lambda(\psi),M)\Big)\Big].
\end{aligned}\tag{40}$$

We define an operator $\mathcal{T}_{\varepsilon,\eta}$ on $C_b([0,T]\times(\mathbf{C}\oplus\mathbf{B}))$ as follows,

$$\begin{aligned}
\mathcal{T}_{\varepsilon,\eta}W(t,\psi)&\equiv\frac{1}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r)+(1-\rho(t))}\Big[\rho(t)\Big(\frac{1}{\varepsilon}W(t,\tilde{\psi}_\varepsilon)\\
&+\frac{W(t+\eta,\psi)}{\eta}+\frac{1}{2}\frac{W(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})+W(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\\
&+\frac{W(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\Big)+(1-\rho(t))\Big(\min(\lambda(\psi),M)\Big)\Big].
\end{aligned}\tag{41}$$

Lemma 5.4 *For each ε and η, $\mathcal{T}_{\varepsilon,\eta}$ is a contraction on $C_b([0,T)\times(\mathbf{C}\oplus\mathbf{B}))$.*

Proof. To prove that $\mathcal{T}_{\varepsilon,\eta}$ is a contraction, we need to show that there exists a constant $0 < \beta < 1$ such that

$$\|\mathcal{T}_{\varepsilon,\eta}W_1 - \mathcal{T}_{\varepsilon,\eta}W_2\| \leq \beta\|W_1 - W_2\| \quad \text{for all} \quad W_1, W_2 \in C_b([0,T] \times (\mathbf{C} \oplus \mathbf{B})),$$

where $\|\cdot\|$ is the sup norm define by (5). Let us define $c_{\varepsilon,\eta} : [0,T] \to \mathbb{R}$ by

$$c_{\varepsilon,\eta}(t) = \rho(t)\left(\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2} + r\right) + (1 - \rho(t)).$$

Note that

$$\begin{aligned}
&|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi) - \mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)| \\
&\leq \Bigg[\frac{1}{c_{\varepsilon,\eta}(t)}\Bigg|\rho(t)\Bigg(\frac{1}{\varepsilon}W_1(t,\tilde{\psi}_\varepsilon) + \frac{W_1(t,\psi + \varepsilon r\psi(0)\mathbf{1}_{\{0\}}))}{\varepsilon} + \frac{W_1(t+\eta,\psi)}{\eta} \\
&\quad + \frac{1}{2}\frac{W_1(t,\psi + \varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}}) + W_1(t,\psi - \varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\Bigg) \\
&\quad - \rho(t)\Bigg(\frac{1}{\varepsilon}W_2(t,\tilde{\psi}_\varepsilon) + \frac{W_2(t,\psi + \varepsilon r\psi(0)\mathbf{1}_{\{0\}}))}{\varepsilon} + \frac{W_2(t+\eta,\psi)}{\eta} \\
&\quad + \frac{1}{2}\frac{W_2(t,\psi + \varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}}) + W_2(t,\psi - \varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2}\Bigg)\Bigg|\Bigg].
\end{aligned}$$

This implies that for all t, ψ,

$$\Big|\mathcal{T}_{\varepsilon,\eta}W_1(t,\psi) - \mathcal{T}_{\varepsilon,\eta}W_2(t,\psi)\Big| \leq \left[\frac{\rho(t)\left(\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2}\right)}{c_{\varepsilon,\eta}(t)}\right]\|W_1 - W_2\|.$$

In addition, note that, for all $t \in [0,T)$,

$$\begin{aligned}
\frac{\rho(t)(\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2})}{c_{\varepsilon,\eta}(t)} &= \frac{\rho(t)(\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2})}{\rho(t)(\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2} + r) + (1 - \rho(t))} \\
&= \frac{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2}}{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2} + r} < 1.
\end{aligned}$$

Take

$$\beta_{\varepsilon,\eta} = \frac{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2}}{\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{1}{\varepsilon^2} + r}.$$

Therefore,

$$\|\mathcal{T}_{\varepsilon,\eta}W_1 - \mathcal{T}_{\varepsilon,\eta}W_2\| \leq \beta_{\varepsilon,\eta}\|W_1 - W_2\|.$$

□

Definition 5.5 The scheme S_M is said to be **stable** if for every $\varepsilon, \eta \in (0,1)$, there exists a bounded solution $W_{\varepsilon,\eta}$ to the equation

$$S_M(\varepsilon, \eta, t, \psi, W(t,\psi), W) = 0, \tag{42}$$

with the bound independent of ε, and η.

Remark 5.6 By the Banach fixed point theorem, the strict contraction $\mathcal{T}_{\varepsilon,\eta}$ has a unique fixed point that we denote by $\widetilde{W}^M_{\varepsilon,\eta} \in C([0,T)\times(\mathbf{C}\oplus\mathbf{B}))_b$. We extend $\widetilde{W}^M_{\varepsilon,\eta}$ on $[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ and denote $W^M_{\varepsilon,\eta}$ its extension. For all $\psi \in \mathbf{C}\oplus\mathbf{B}$, we have

$$W^M_{\varepsilon,\eta}(t,\psi) = \begin{cases} \widetilde{W}^M_{\varepsilon,\eta}(t,\psi), & \text{if} \quad t \in [0,T), \\ \min(\lambda(\psi), M), & \text{if} \quad t = T. \end{cases}$$

Moreover the function $W^M_{\varepsilon,\eta}$ satisfies equation (40), thus it solves equation (42). Given any function $\widetilde{W}_0 \in C([0,T)\times(\mathbf{C}\oplus\mathbf{B}))_b$ with $\|\widetilde{W}_0\| < M$, we construct a sequence as follows, $\widetilde{W}_{n+1} = \mathcal{T}_{\varepsilon,\eta}\widetilde{W}_n$ for $n \geq 0$. It is clear that

$$\lim_{n\to\infty} \widetilde{W}_n = \widetilde{W}^M_{\varepsilon,\eta}.$$

By induction we have

$$\begin{aligned} & \widetilde{W}_{n+1}(t,\psi) \\ = \; & \frac{1}{c_{\varepsilon,\eta}(t)}\Bigg[\rho(t)\Bigg(\frac{1}{\varepsilon}\widetilde{W}_n(t,\tilde{\psi}_\varepsilon) + \frac{\widetilde{W}_n(t+\eta,\psi)}{\eta} + \frac{\widetilde{W}_n(t,\psi+\varepsilon\psi(0)r\mathbf{1}_{\{0\}})}{\varepsilon}\Bigg) \\ & + \frac{1}{2}\frac{\widetilde{W}_n(t,\psi+\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}}) + \widetilde{W}_n(t,\psi-\varepsilon\psi(0)g(\psi)\mathbf{1}_{\{0\}})}{\varepsilon^2} \\ & + (1-\rho(t))\Bigg(\min(\lambda(\psi), M)\Bigg)\Bigg] \\ \leq \; & \frac{\rho(t)M(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}) + (1-\rho(t))\min(\lambda(\psi),M)}{\rho(t)(\frac{2}{\varepsilon}+\frac{1}{\eta}+\frac{1}{\varepsilon^2}+r) + (1-\rho(t))} \\ \leq \; & M. \end{aligned} \tag{43}$$

Therefore $\|\widetilde{W}^M_{\varepsilon,\eta}\| \leq M$, thus $\|W^M_{\varepsilon,\eta}\| \leq M$. This implies that our scheme S_M is stable.

Theorem 5.7 *Let $W^M_{\varepsilon,\eta}$ denote the solution to* (42)*. Then, as $(\varepsilon,\eta) \to 0$, the sequence $W^M_{\varepsilon,\eta}$ converges uniformly on $[0,T]\times\mathbf{C}$ to the unique viscosity solution V_M of* (31).

Proof. Define

$$\begin{aligned} W_M^*(t,\psi) &= \limsup_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W_{\varepsilon,\eta}^M(\tau,\phi) \\ W_{*M}(t,\psi) &= \liminf_{\tau\to t,\phi\to\psi,\varepsilon\downarrow 0,\eta\downarrow 0} W_{\varepsilon,\eta}^M(\tau,\phi). \end{aligned} \tag{44}$$

We claim that W_M^* and W_{*M} are sub- and supersolutions of (31), respectively. To prove this claim, we only consider the case for W_M^*. The argument for that of W_{*M} is similar. We want to show:

$$\mathcal{H}(t,\psi,W_M^*,\overline{D\Gamma(t,\psi)},\overline{D^2\Gamma(t,\psi)})\le 0,$$

for any test function $\Gamma\in C_{lip}^{1,2}([0,T]\times(\mathbf{C}\oplus\mathbf{B}))\cap\mathcal{D}(\mathcal{S})$ such that (t,ψ) is a strictly local maximum of $W_M^*(\tau,\phi)-\Gamma(\tau,\phi)$. Without loss of generality, we may assume that $W_M^*\le\Gamma$ and $W_M^*(t,\psi)=\Gamma(t,\psi)$ and because of the stability of our scheme we can also assume that $\Gamma\ge 2\sup_{\varepsilon,\eta}\|W_{\varepsilon,\eta}^M\|$ outside of the ball $B((t,\psi),d)$ where $d>0$ is such that

$$W_M^*(\tau,\phi)-\Gamma(\tau,\phi)\le 0=W_M^*(t,\psi)-\Phi(t,\psi) \text{ for } (\tau,\phi)\in B((t,\psi),d).$$

This implies that there exist sequences $\varepsilon_n>0$, $\eta_n>0$, and $(\tau_n,\phi_n)\in[0,T]\times(\mathbf{C}\oplus\mathbf{B})$ such that as $n\to\infty$ we have

$$\begin{aligned} &\varepsilon_n\to 0,\ \ \eta_n\to 0,\ \ \tau_n\to t,\ \ \phi_n\to\psi,\ \ W_{\varepsilon_n,\eta_n}^M(\tau_n,\phi_n)\to W_M^*(t,\psi), \\ &\text{and } (\tau_n,\phi_n) \text{ is a global maximum } W_{\varepsilon_n,\eta_n}^M-\Gamma. \end{aligned} \tag{45}$$

Denote $\alpha_n=W_{\varepsilon_n,\eta_n}^M(\tau_n,\phi_n)-\Gamma(\tau_n,\phi_n)$. Obviously $\alpha_n\to 0$ and

$$W_{\varepsilon_n,\eta_n}^M(\tau,\phi)\le\Gamma(\tau,\phi)+\alpha_n \text{ for all } (\tau,\phi)\in[0,T]\times(\mathbf{C}\oplus\mathbf{B}). \tag{46}$$

We know that

$$S_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W_{\varepsilon_n,\eta_n}^M(\tau_n,\phi_n),W_{\varepsilon_n,\eta_n}^M)=0.$$

The monotonicity of S_M and (46) implies

$$\begin{aligned} &\quad S_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,\Gamma(\tau_n,\phi_n)+\alpha_n,\Gamma+\alpha_n) \\ &\le S_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W_{\varepsilon_n,\eta_n}^M(\tau_n,\phi_n),W_{\varepsilon_n,\eta_n}^M) \\ &= 0. \end{aligned} \tag{47}$$

Therefore,

$$\lim_{n\to\infty}\frac{S_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W_{\varepsilon_n,\eta_n}^M(\tau_n,\phi_n),W_{\varepsilon_n,\eta_n}^M)}{\varepsilon_n}\le 0,$$

so we have

$$\begin{aligned} &\mathcal{H}(t,\psi,W_M^*,\overline{D\Gamma(t,\psi)},\overline{D^2\Gamma(t,\psi)}) \\ &\quad=\lim_{n\to\infty}\frac{S_M(\varepsilon_n,\eta_n,\tau_n,\phi_n,W_{\varepsilon_n,\eta_n}^M(\tau_n,\phi_n),W_{\varepsilon_n,\eta_n}^M)}{\varepsilon_n}\le 0. \end{aligned} \tag{48}$$

This proves that W_M^* is a viscosity subsolution of (31). Similarly we can prove that W_{*M} is a viscosity supersolution. By virtue of Theorem 4.4, we can get that

$$W_{*M}(t,\psi) \geq W_M^*(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}. \tag{49}$$

On the other hand, by the definition of W_{*M}, W_M^*, it is easy to see that

$$W_{*M}(t,\psi) \leq W_M^*(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Combined with (49), the above inequality implies

$$W_{*M}(t,\psi) = W_M^*(t,\psi), \quad \forall (t,\psi) \in [0,T] \times \mathbf{C}.$$

Since W_{*M} is a viscosity supersolution and W_M^* is a viscosity subsolution, they are also viscosity solutions of (31). Now, using the uniqueness of the viscosity solution (31), we see that $V_M = W_M^* = W_{*M}$. Therefore, we conclude that the sequence $(W_{\varepsilon,\eta}^M)_{\varepsilon,\eta}$ converges uniformly to V_M as desired. □

6 The Algorithm

Based on the results obtained in the last section, we can construct the computational algorithm to obtain a numerical solution. For example, one algorithm can be like the following:

Step 0. Choose any function $W^{(0)} \in C([0,T] \times \mathbf{C} \oplus \mathbf{B})_b$;

Step 1. Pick the starting values for $\varepsilon(1), \eta(1)$. For example, we can choose $\varepsilon(1) = 10^{-2}, \eta(1) = 10^{-3}$;

Step 2. For the given $\varepsilon, \eta > 0$, compute the function

$$W_{\varepsilon(1),\eta(1)}^{(1)} \in C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$$

by the following formula

$$W_{\varepsilon(1),\eta(1)}^{(1)} = \mathcal{T}_{\varepsilon(1),\eta(1)} W^{(0)},$$

where $\mathcal{T}_{\varepsilon(1),\eta(1)}$, which is defined on $C_b([0,T] \times \mathbf{C} \oplus \mathbf{B})$, is given by (41);

Step 3. Repeat Step 2 for $i = 2, 3, \cdots$ using

$$W_{\varepsilon(1),\eta(1)}^{(i)} = \mathcal{T}_{\varepsilon(1),\eta(1)} W_{\varepsilon(1),\eta(1)}^{(i-1)},$$

where $\mathcal{T}_{\varepsilon(1),\eta(1)}$, which is defined on $C_b([0,T]\times \mathbf{C}\oplus\mathbf{B})$, is given by (41). Stop the iteration when

$$\|W^{i+1}_{\varepsilon(1),\eta(1)}(t,\psi) - W^{i}_{\varepsilon(1),\eta(1)}(t,\psi)\| \le \delta_1,$$

where δ_1 is a preselected number which is small enough to achieve the accuracy we want. Denote the final solution by $W_{\varepsilon(1),\eta(1)}(t,\psi)$.

Step 4. Choose two sequences of $\varepsilon(k)$ and $\eta(k)$, such that

$$\lim_{k\to\infty}\varepsilon(k) = \lim_{k\to\infty}\eta(k) = 0.$$

For example, we may choose $\varepsilon(k) = \eta(k) = 10^{-(2+k)}$. Now repeat Step 2 and Step 3 for each $\varepsilon(k),\eta(k)$ until

$$\|W_{\varepsilon(k+1),\eta(k+1)}(t,\psi) - W_{\varepsilon(k),\eta(k)}(t,\psi)\| \le \delta_2,$$

where δ_2 is chosen to obtain the expected accuracy.

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