Let us write down equation number \( k \) from the system \( Ax = f \):

\[
a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n = f_k.
\]

Taking into account that \(|x_k| \geq |x_j|\) for \( j = 1, 2, \ldots, n \), we arrive at the following estimate:

\[
|f_k| = \left| \sum_j a_{kj} x_j \right| \geq |a_{kk}| |x_k| - \left( \sum_{j \neq k} |a_{kj}| \right) |x_k| = \left( |a_{kk}| - \sum_{j \neq k} |a_{kj}| \right) |x_k| \geq \delta |x_k|.
\]

Consequently, \(|x_k| \leq |f_k|/\delta\). On the other hand, \(|x_k| = \max_j |x_j| = \|x\|_\infty\) and \(|f_k| \leq \max_i |f_i| = \|f\|_\infty\). Therefore,

\[
\|x\|_\infty \leq \frac{1}{\delta} \|f\|_\infty.
\]  

In particular, estimate (5.42) means that if \( f = 0 \in \mathbb{L} \) (e.g., \( \mathbb{L} = \mathbb{R}^n \) or \( \mathbb{L} = \mathbb{C}^n \)), then \( \|x\|_\infty = 0 \), and consequently, the homogeneous system \( Ax = 0 \) only has a trivial solution \( x = 0 \). As such, the inhomogeneous system \( Ax = f \) has a unique solution for every \( f \in \mathbb{L} \). In other words, the inverse matrix \( A^{-1} \) exists.

Estimate (5.42) also implies that for any \( f \in \mathbb{L}, f \neq 0 \), the following estimate holds for \( x = A^{-1}f \):

\[
\|A^{-1}f\|_\infty \leq \frac{1}{\delta} \|f\|_\infty \quad \implies \quad \frac{\|A^{-1}f\|_\infty}{\|f\|_\infty} \leq \frac{1}{\delta},
\]

so that

\[
\|A^{-1}\|_\infty = \max_{f \in \mathbb{L}, f \neq 0} \frac{\|A^{-1}f\|_\infty}{\|f\|_\infty} \leq \frac{1}{\delta}.
\]

**COROLLARY 5.1**

Let \( A \) be a matrix with diagonal dominance of magnitude \( \delta > 0 \). Then,

\[
\mu_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty \leq \frac{1}{\delta} \|A\|_\infty.
\]  

(5.43)

The proof is obtained as an immediate implication of the result of Theorem 5.5.

**Exercises**

1. Prove that the condition numbers \( \mu_\infty(A) \) and \( \mu_1(A) \) of the matrix \( A \) will not change after any permutation of rows and/or columns.
2. Prove that for a square matrix $A$ and its transpose $A^T$, the following equalities hold: 
\[ \mu_{\infty}(A) = \mu_1(A^T), \mu_1(A) = \mu_{\infty}(A^T). \]

3. Show that the condition number of the operator $A$ does not change if the operator is multiplied by an arbitrary non-zero real number.

4. Let $L$ be a Euclidean space, and let $A : L \mapsto L$. Show that the condition number $\mu_B(A)$ of the operator $A$ does not change if the operator is multiplied by an arbitrary non-zero real number.

5. Prove that $\mu_B(A) = \mu_B(A^*)$, where $A^*$ is the operator adjoint to $A$ in the sense of the scalar product $[x,y]_B$.

6. Let $A$ be a non-singular matrix, $\det A \neq 0$. Multiply one row of the matrix $A$ by some scalar $\alpha$, and denote the new matrix by $A_\alpha$. Show that $\mu(A_\alpha) \to \infty$ as $\alpha \to \infty$.

7. Prove that for any linear operator $A : L \mapsto L$: 
\[ \mu_B(A^*BA) = (\mu_B(A))^2, \]
where $A^*$ is the operator adjoint to $A$ in the sense of the scalar product $[x,y]_B$.

8. Let $A = A^* > 0$ and $B = B^* > 0$ in the sense of some scalar product introduced on the linear space $L$. Let the following inequalities hold for every $x \in L$: 
\[ \gamma_1(Bx,x) \leq (Ax,x) \leq \gamma_2(Bx,x), \]
where $\gamma_1 > 0$ and $\gamma_2 > 0$ are two real numbers. Consider the operator $C = B^{-1}A$ and prove that the condition number $\mu_B(C)$ satisfies the estimate: 
\[ \mu_B(C) \leq \frac{\gamma_2}{\gamma_1}. \]

**Remark.** We will solve this problem in Section 6.1.4 as it has numerous applications.

### 5.4 Gaussian Elimination and Its Tri-Diagonal Version

We will describe both the standard Gaussian elimination algorithm and the Gaussian elimination with pivoting, as they apply to solving an $n \times n$ system of linear algebraic equations in its canonical form:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= f_1, \\
  & \quad \cdots \\
  a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= f_n.
\end{align*}
\]

(5.44)

Recall that the Gaussian elimination procedures belong to the class of direct methods, i.e., they produce the exact solution of system (5.44) after a finite number of arithmetic operations.