The key point, of course, is to obtain a convenient expression for the integral on the right-hand side of formula (4.20) via the function values \( f(x_m), m = 0, 1, 2, \ldots, m \), sampled on the Gauss grid (4.17). It is precisely the introduction of the weight \( w(x) = 1/\sqrt{1-x^2} \) into the integral (4.20) that enables a particularly straightforward integration of the interpolating polynomial \( P_n(x, f) \) over \([-1, 1]\).

**LEMMA 4.2**

Let the function \( f = f(x) \) be defined on \([-1, 1]\), and let \( P_n(x, f) \) be its algebraic interpolating polynomial on the Chebyshev-Gauss grid (4.17). Then,

\[
\int_{-1}^{1} \frac{P_n(x, f)}{\sqrt{1-x^2}} \, dx = \frac{\pi}{n+1} \sum_{m=0}^{n} f(x_m). \tag{4.21}
\]

**PROOF** Recall that \( P_n(x, f) \) is an interpolating polynomial of degree no higher than \( n \) built for the function \( f(x) \) on the Gauss grid (4.17). The grid has a total of \( n + 1 \) nodes, and the polynomial is unique. As shown in Section 3.2.3, see formulae (3.62) and (3.63) on page 76, \( P_n(x, f) \) can be represented as:

\[
P_n(x, f) = \sum_{k=0}^{n} a_k T_k(x),
\]

where \( T_k(x) \) are Chebyshev polynomials of degree \( k \), and the coefficients \( a_k \) are given by:

\[
a_0 = \frac{1}{n+1} \sum_{m=0}^{n} f_m T_0(x_m) \quad \text{and} \quad a_k = \frac{2}{n+1} \sum_{m=0}^{n} f_m T_k(x_m), \quad k = 1, \ldots, n.
\]

Accordingly, it will be sufficient to show that equality (4.21) holds for all individual \( T_k(x) \), \( k = 0, 1, 2, \ldots, n \):

\[
\int_{-1}^{1} \frac{T_k(x)}{\sqrt{1-x^2}} \, dx = \frac{\pi}{n+1} \sum_{m=0}^{n} T_k(x_m). \tag{4.22}
\]

Let \( k = 0 \), then \( T_0(x) \equiv 1 \). In this case Lemma 4.1 implies that the left-hand side of (4.22) is equal to \( \pi \), and the right-hand side also appears equal to \( \pi \) by direct substitution. For \( k > 0 \), orthogonality (4.19) means that the left-hand side of (4.22) is equal to zero. To prove that the right-hand side is zero as well, we employ formula (3.22). The range of summation in (3.22) is from \( m = 0 \) to \( m = N - 1 \), where \( N = 2(n + 1) \). As, however, cosine is an even function, the same result will hold for half the summation range and any \( k = 1, 2, \ldots, 2n + 1 \):

\[
\sum_{m=0}^{n} T_k(x_m) = \sum_{m=0}^{n} \cos \left( k \arccos \left( \cos \frac{2m+1}{2(n+1)} \right) \right) = \sum_{m=0}^{n} \cos \left( k \frac{2m+1}{2(n+1)} \right) = \sum_{m=0}^{n} \cos \left( k \frac{2m}{2(n+1)} + k \frac{\pi}{2(n+1)} \right) = 0.
\]
This implies, in particular, that the right-hand side of (4.22) is zero for all 
\( k = 1, 2, \ldots, n \). Thus, we have established equality (4.21).

Lemma 4.21 allows us to recast the approximate expression (4.20) for the integral
\[ \int_{-1}^{1} f(x)w(x)dx \]
as follows:
\[ \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \approx \pi \frac{1}{n+1} \sum_{m=0}^{n} f(x_m). \] (4.23)

Formula (4.23) is known as the Gaussian quadrature formula with the weight
\( w(x) = \frac{1}{\sqrt{1-x^2}} \) on the Chebyshev-Gauss grid (4.17). It has a particularly simple structure, which is very convenient for implementation. In practice, if we need to evaluate the integral \( \int_{-1}^{1} f(x)dx \) for a given \( f(x) \) with no weight, we introduce a new function
\( g(x) = f(x)\sqrt{1-x^2} \) and then rewrite formula (4.23) as:
\[ \int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}dx \approx \pi \frac{1}{n+1} \sum_{m=0}^{n} g(x_m). \]

A key advantage of the Gaussian quadrature (4.23) compared to the quadrature
formulae studied previously in Section 4.1 is that the Gaussian quadrature does not
get saturated by smoothness. Indeed, according to the following theorem (see also
Remark 4.2 right after the proof of Theorem 4.5), the integration error automatically
adjusts to the regularity of the integrand.

**THEOREM 4.5**
Let the function \( f = f(x) \) be defined for \(-1 \leq x \leq 1\); let it have continuous
derivatives up to the order \( r > 0 \), and a square integrable derivative of order
\( r+1 \):
\[ \int_{-1}^{1} [f^{(r+1)}(x)]^2 dx < \infty. \]

Then, the error of the Gaussian quadrature (4.23) can be estimated as:
\[ \left| \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx - \pi \frac{1}{n+1} \sum_{m=0}^{n} f(x_m) \right| \leq \pi \frac{\zeta_n}{n^{r-1/2}}, \] (4.24)

where \( \zeta_n = o(1) \) as \( n \rightarrow \infty \).

**PROOF** The proof of inequality (4.24) is based on the error estimate
(3.65) obtained in Section 3.2.4 (see page 77) for the Chebyshev algebraic in-
terpolation. Namely, let \( R_n(x) = f(x) - P_n(x, f) \). Then, under the assumptions