where each $Q_{2s+1}(x,k)$ is a polynomial of degree no greater than $2s+1$ that satisfies the relations:

$$\left.\frac{d^m Q_{2s+1}(x,k)}{dx^m}\right|_{x=x_k} = \left.\frac{d^m P_s(x,f_{kj})}{dx^m}\right|_{x=x_k}, \qquad m=0,1,2,\dots,s, \tag{2.43}$$

$$\left.\frac{d^m Q_{2s+1}(x,k)}{dx^m}\right|_{x=x_{k+1}} = \left.\frac{d^m P_s(x,f_{k+1,j})}{dx^m}\right|_{x=x_{k+1}}, \qquad m=0,1,2,\dots,s. \tag{2.44}$$

THEOREM 2.8
The polynomial $Q_{2s+1}(x,k)$ of degree no greater than $2s+1$ defined by means of equalities (2.43) and (2.44) exists and is unique.

PROOF Let $Q_{2s+1}(x,k) = c_{0,k} + c_{1,k}x + \ldots + c_{2s+1,k}x^{2s+1}$. Then, relations (2.43) and (2.44) together can be considered as a system of $2s+2$ linear algebraic equations with respect to the $2s+2$ unknown coefficients $c_{0,k}$, $c_{1,k}$, ..., $c_{2s+1,k}$. Let us analyze its homogeneous counterpart obtained by replacing all the right-hand sides of equalities (2.43) and (2.44) by zeros. Homogeneity of (2.43), (2.44) clearly implies that the polynomial $Q_{2s+1}(x,k)$ has a root of multiplicity $s+1$ at $x=x_k$ and another root of multiplicity $s+1$ at $x=x_{k+1}$. In other words, $Q_{2s+1}(x,k)$ has a total of $2s+2$ roots counting their multiplicities. This is only possible if $Q_{2s+1}(x,k)\equiv 0$, because $Q_{2s+1}(x,k)$ is a polynomial of degree no greater than $2s+1$. Consequently, $c_{0,k} = c_{1,k} = \ldots = c_{2s+1,k} = 0$, and we conclude that the homogeneous counterpart of the linear algebraic system (2.43), (2.44) may only have a trivial solution. As such, the original inhomogeneous system (2.43), (2.44) itself will have a unique solution for any choice of its right-hand sides. ∎

THEOREM 2.9
Let $f(x)$ be a polynomial of degree no greater than s. Then, the interpolant $\varphi(x,s)$ coincides with this polynomial.

PROOF We will prove the identity $\varphi(x,s)\equiv f(x)$ on the interval $[x_k,x_{k+1}]$ for an arbitrary k, i.e., for all x in between any two neighboring interpolation nodes. In other words, we will prove that $Q_{2s+1}(x,k)\equiv f(x)$. Due to the uniqueness of the interpolating polynomial, we have $P_s(x,f_{kj})\equiv P_s(x,f_{k+1,j})\equiv f(x)$. Then, clearly, the polynomial $f(x)$ solves system (2.43), (2.44). ∎

THEOREM 2.10
The piecewise polynomial interpolating function $\varphi(x,s)$ defined by equalities (2.42) assumes the given values $f(x_k)$ at the interpolation nodes x_k, $k=0,\pm1,\ldots$. Moreover, $\varphi(x,s)$ has a continuous derivative of order s everywhere on its domain.