where each $Q_{2s+1}(x,k)$ is a polynomial of degree no greater than $2s + 1$ that satisfies the relations:

$$
\frac{d^m Q_{2s+1}(x,k)}{dx^m} \bigg|_{x=x_k} = \frac{d^m P_s(x, f_{kj})}{dx^m} \bigg|_{x=x_k}, \quad m = 0, 1, 2, \ldots, s, \quad (2.43)
$$

$$
\frac{d^m Q_{2s+1}(x,k)}{dx^m} \bigg|_{x=x_{k+1}} = \frac{d^m P_s(x, f_{k+1,j})}{dx^m} \bigg|_{x=x_{k+1}}, \quad m = 0, 1, 2, \ldots, s. \quad (2.44)
$$

**THEOREM 2.8**

The polynomial $Q_{2s+1}(x,k)$ of degree no greater than $2s + 1$ defined by means of equalities (2.43) and (2.44) exists and is unique.

**PROOF** Let $Q_{2s+1}(x,k) = c_{0,k} + c_{1,k}x + \ldots + c_{2s+1,k}x^{2s+1}$. Then, relations (2.43) and (2.44) together can be considered as a system of $2s+2$ linear algebraic equations with respect to the $2s+2$ unknown coefficients $c_{0,k}$, $c_{1,k}$, ..., $c_{2s+1,k}$. Let us analyze its homogeneous counterpart obtained by replacing all the right-hand sides of equalities (2.43) and (2.44) by zeros. Homogeneity of (2.43), (2.44) clearly implies that the polynomial $Q_{2s+1}(x,k)$ has a root of multiplicity $s+1$ at $x = x_k$ and another root of multiplicity $s+1$ at $x = x_{k+1}$. In other words, $Q_{2s+1}(x,k)$ has a total of $2s+2$ roots counting their multiplicities. This is only possible if $Q_{2s+1}(x,k) \equiv 0$, because $Q_{2s+1}(x,k)$ is a polynomial of degree no greater than $2s + 1$. Consequently, $c_{0,k} = c_{1,k} = \ldots = c_{2s+1,k} = 0$, and we conclude that the homogeneous counterpart of the linear algebraic system (2.43), (2.44) may only have a trivial solution. As such, the original inhomogeneous system (2.43), (2.44) itself will have a unique solution for any choice of its right-hand sides.

**THEOREM 2.9**

Let $f(x)$ be a polynomial of degree no greater than $s$. Then, the interpolant $\varphi(x,s)$ coincides with this polynomial.

**PROOF** We will prove the identity $\varphi(x,s) \equiv f(x)$ on the interval $[x_k, x_{k+1}]$ for an arbitrary $k$, i.e., for all $x$ in between any two neighboring interpolation nodes. In other words, we will prove that $Q_{2s+1}(x,k) \equiv f(x)$. Due to the uniqueness of the interpolating polynomial, we have $P_s(x, f_{kj}) \equiv P_s(x, f_{k+1,j}) \equiv f(x)$. Then, clearly, the polynomial $f(x)$ solves system (2.43), (2.44).

**THEOREM 2.10**

The piecewise polynomial interpolating function $\varphi(x,s)$ defined by equalities (2.42) assumes the given values $f(x_k)$ at the interpolation nodes $x_k$, $k = 0, \pm 1, \ldots$. Moreover, $\varphi(x,s)$ has a continuous derivative of order $s$ everywhere on its domain.