7. Consider Cauchy problem (10.67) for the heat equation, and approximate it with the predictor-corrector scheme designed as follows. The auxiliary grid function $\tilde{u}_m^{p+1/2}$ is to be computed by the implicit method:
\[
\tilde{u}_m^{p+1/2} - u_m^p = \frac{\tilde{u}_m^{p+1/2} - 2u_m^{p+1/2} + \tilde{u}_m^{p+1/2}}{h^2} = 0, \quad m = 0, \pm 1, \pm 2, \ldots,
\]
and the actual solution $u_m^{p+1}$ at $t = t_{p+1}$ is to be computed by the scheme:
\[
\frac{u_m^{p+1} - u_m^p}{\tau} = \frac{\tilde{u}_m^{p+1} - 2u_m^{p+1/2} + \tilde{u}_m^{p+1/2}}{h^2} = 0, \quad u_m^0 = \psi(mh).
\]
Prove that the overall predictor-corrector scheme has accuracy $O(\tau^2 + h^2)$ on the smooth solution $u = u(x,t)$.

8. Consider a modified scheme $L_{h}u^{(h)} = f^{(h)}$ of type (10.47) (modification of the right-hand side only):
\[
L_{h}u^{(h)} = \begin{cases}
\phi(x_m,t_p) + \frac{\tau h}{2}(\phi_t + \phi_x)(x_m,t_p), \\
\psi(x_m),
\end{cases}
\]
and define the coefficients of the operator $L_h$ according to formula (10.55) that corresponds to the Lax-Wendroff method. Show that the resulting scheme approximates problem (10.8) with second order accuracy for an arbitrary sufficiently smooth right-hand side $\phi(x,t)$ (not necessarily zero).

9. Represent scheme (10.47) with $\phi(x,t) = 0$ in the form:
\[
u_m^{p+1} = b_{-1}u_{m-1}^p + b_0u_m^p + b_1u_{m+1}^p,
\]
where $b_{-1} = -a_{-1}/\alpha^0$, $b_0 = a_0/\alpha^0$, and $b_1 = -a_1/\alpha^0$, see Section 10.2.3. Scheme (10.68) is said to be monotone if $b_j \geq 0$, $j = -1,0,1$. Adopting the terminology of Section 10.2.3, let the primary constraint be the order of accuracy of at least $O(h)$, and the secondary constraint be monotonicity of the scheme. In the three-dimensional space of vectors $\{b_{-1}, b_0, b_1\}$, describe the set $M_0 = M_{ps} \cap M_{sec}$ (in this case, the distance between the sets $M_{ps}$ and $M_{sec}$ is zero).

\textbf{Answer.} If $r = \tau/h > 1$, then $M_0 = \emptyset$. If $r = 1$, then $M_0 = \{(0,0,1)\}$. If $0 < r < 1$, then $M_0$ is the interval with the endpoints: $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $(0, 1-r, r)$.

10. Prove that the monotone schemes defined the same way as in Exercise 9 for $\phi = 0$ and $j = j_{\text{left}}, \ldots, j_{\text{right}}$ satisfy the maximum principle from page 315, i.e., that the maximum of the corresponding difference solution will not increase as the time elapses.

11. \textbf{(Godunov theorem)} Prove that no one-step explicit monotone scheme may have accuracy higher than $O(h)$ on smooth solutions of the differential equation $u_t + cu_x = 0$.

\textbf{Hint.} One version of the proof, due to Harten, Hyman, and Lax, can be found, e.g., in [Str04, pages 71–72]. Note also that there are exceptions, but they are all trivial. For example, the first order upwind scheme with $r = 1$ has zero error on the exact solution $u = u(x-t)$ of $u_t + cu_x = 0$. 

}\]