

equations with respect to the grid function u_m^{p+1} of the argument m . Nonetheless, it will later be shown (see Examples 6 and 7 of Section 10.3 and also Section 10.6) that the implicit scheme $\mathbf{L}^{(2)}u^{(h)} = f^{(h)}$ is often more convenient for practical computations compared to the explicit scheme $\mathbf{L}^{(1)}u^{(h)} = f^{(h)}$. The stencils of both schemes are schematically depicted in Figure 10.3.

When $\tau = rh^2$, $r = \text{const}$, both schemes have second order accuracy. For the explicit scheme (10.37), this result is a part of Exercise 3 of Section 10.1. Let us therefore prove second order accuracy for the implicit scheme (10.38).

Using formulae (10.35), we can write:

$$\mathbf{L}_h^{(2)}[u]_h = \begin{cases} \frac{\partial u(x_m, t_{p+1})}{\partial t} - \frac{\partial^2 u(x_m, t_{p+1})}{\partial x^2} \\ - \frac{\tau}{2} \frac{\partial^2 u(x_m, t_{p+1})}{\partial t^2} - \frac{h^2}{12} \frac{\partial^4 u(x_m, t_{p+1})}{\partial x^4} + o(\tau + h^2), \\ u(x_m, 0). \end{cases}$$

Consequently, considering that $\tau = rh^2$, we have:

$$\mathbf{L}_h^{(2)}[u]_h = \begin{cases} \varphi(mh, (p+1)\tau) + \mathcal{O}(h^2), \\ \psi(mh) + 0. \end{cases}$$

Therefore, for the truncation error $\delta f^{(h)} = \mathbf{L}_h^{(2)}[u]_h - f^{(h)}$ we obtain:

$$\delta f^{(h)} = \begin{cases} \mathcal{O}(h^2), \\ 0, \end{cases}$$

which obviously implies that

$$\|\delta f^{(h)}\|_{F_h} = \mathcal{O}(h^2).$$

Example 3

Let us now consider a simple finite-difference scheme that would approximate the inhomogeneous Dirichlet problem of type (5.7) for the Poisson equation on the square $D = \{(x, y) | 0 < x < 1, 0 < y < 1\}$ with the boundary $\Gamma = \partial D$:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \varphi(x, y), & (x, y) \in D, \\ u|_{\Gamma} &= \psi(x, y), & (x, y) \in \Gamma. \end{aligned}$$

As we did previously in Section 5.1.3, we will use a uniform Cartesian grid in D with size h in either coordinate direction; and we will assume that $M = 1/h$ is an integer:

$$D_h = \{(x_m, y_n) | x_m = mh, y_n = nh, m = 0, 1, \dots, M, n = 0, 1, \dots, M\}.$$