A minimum norm weak (generalized) solution of the overdetermined system (7.27) is the vector \( \hat{x} \in \mathbb{R}^n \) that minimizes \( \Phi(x) \), i.e., \( \forall x \in \mathbb{R}^n, \Phi(x) \leq \Phi(\hat{x}) \), and also such that if there is another \( x \in \mathbb{R}^n \), \( \Phi(x) = \Phi(\hat{x}) \), then \( \|x\| \geq \|\hat{x}\| \).

Note that the minimum norm weak solution introduced according to Definition 7.3 may exhibit strong sensitivity to the perturbations of the matrix \( A \) in the case when these perturbations change the rank of the matrix, see the example given in Exercise 2 after the section.

REMARK 7.6 Definition 7.3 can also be applied to the case of a full rank matrix \( A \), \( \text{rank} A = n \). Then it reduces to Definition 7.1 (for \( B = I \)), because according to Theorem 7.2 a unique least squares weak solution exists for a full rank overdetermined system, and consequently, the Euclidean norm of this solution is minimum.

THEOREM 7.3

Let \( A \) be an \( m \times n \) matrix with real entries, \( m \geq n \), and \( \text{rank} A = r < n \). There is a unique weak solution of system (7.27) in the sense of Definition 7.3. This solution is given by the formula:

\[
\hat{x} = A^+ f,
\]

where \( A^+ \) is the Moore-Penrose pseudoinverse of \( A \) introduced in Definition 7.2.

PROOF

Using singular value decomposition, represent the system matrix of (7.27) in the form: \( A = U \Sigma W^* \). Also define \( y = W^* x \). Then, according to formula (7.32), we can write:

\[
\Phi(x) = (U\Sigma W^* x - f, U\Sigma W^* x - f)(m) = (U\Sigma y - f, U\Sigma y - f)(m)
\]

\[
= (\Sigma y - U^* f, \Sigma y - U^* f)(m) = \|\Sigma y - U^* f\|_2^2,
\]

and we need to find the vector \( \hat{y} \in \mathbb{R}^n \) such that \( \forall y \in \mathbb{R}^n, \|\Sigma \hat{y} - U^* f\|_2^2 \leq \|\Sigma y - U^* f\|_2^2 \). This vector \( \hat{y} \) must also have a minimum Euclidean norm, because the matrix \( W \) is orthogonal and since \( y = W^* x \) we have \( \|y\|_2 = \|x\|_2 \).

Next, recall that as \( \text{rank} A = r \), the matrix \( A \) has precisely \( r \) non-zero singular values \( \sigma_i \). Then we have:

\[
\|\Sigma y - U^* f\|_2^2 = \sum_{i=1}^r |\sigma_i y_i - (U^* f)_i|^2 + \sum_{i=r+1}^m |(U^* f)_i|^2,
\]

expression (7.34) attains its minimum value when the first sum on the right-hand side is equal to zero, because the second sum simply does not depend