7.3 Rank Deficient Systems. Singular Value Decomposition

In this section, we introduce and study the concept of a weak solution for the linear system:

\[ Ax = f, \quad x \in \mathbb{R}^n, \quad f \in \mathbb{R}^m, \quad (7.27) \]

where \( A \) is an \( m \times n \) matrix with real entries, \( n \leq m \), and with linearly dependent columns, i.e., \( \text{rank} A = r < n \). In this case, the matrix \( A \) is said to be rank deficient, and there are non-trivial linear combinations of the columns \( a_k, k = 1, 2, \ldots, n \), of the matrix \( A \) that are equal to zero: \( \xi_1 a_1 + \xi_2 a_2 + \ldots + \xi_n a_n = 0 \), while \( \sum_{k=1}^{n} |\xi_k|^2 \neq 0 \). Alternatively, we say that the matrix \( A \) has a non-empty kernel:

\[ \mathbb{R}^n \ni x_B = \arg \min_{x \in \mathbb{R}^n} \Phi(x), \quad (7.13) \]

where the function \( \Phi(x) \) is defined by formula (7.13). Then, clearly, \( \forall \xi \in \text{Ker} A : \Phi(x_B + \xi) = \Phi(x_B) \). In terms of matrices, one can show that the matrix \( C = A^* A \) of system (7.14) will be singular in this case.

Recall that when we introduced the original definition of a weak solution there were many alternatives. We could use different weight matrices \( B \) for the function \( \Phi(x) \) of (7.13), and we did not necessarily even have to choose the function \( \Phi \) quadratic, see (7.7). The resulting generalized solution was obviously determined by the choice of \( \Phi \), and it changed when a different \( \Phi \) was selected. Likewise, when we need to extend Definition 7.1 for the case of rank deficient matrices, there may be alternative strategies. A particular approach based on selecting the vector \( x \) with the minimum Euclidean length (i.e., norm) of its own is described in Section 7.3.2. There are other approaches that will yield different weak solutions. To implement the strategy based on minimizing the Euclidean length among all those vectors that minimize \( \Phi \), we need the apparatus of singular value decomposition, see Section 7.3.1.

7.3.1 Singular Value Decomposition and Moore-Penrose Pseudoinverse

In this section, we only provide a brief summary of the results from linear algebra; further detail can be found, e.g., in [HJ85, Chapter 7]. Let \( A \) be a rectangular \( m \times n \) matrix with complex entries. There exist two unitary matrices, \( U \) of dimension \( m \times m \) and \( W \) of dimension \( n \times n \), such that

\[ U^* A W = \Sigma \equiv \text{diag}_{m \times n}(\sigma_1, \sigma_2, \ldots, \sigma_k), \quad k = \min(m, n), \]

\[ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq 0. \quad (7.28) \]