5.7.2 Representation of Solution as a Finite Fourier Series

By direct calculation one can easily make sure\(^6\) that the following equalities hold:

\[
-\Delta^{(h)} \psi^{(k,l)} = \frac{4}{h^2} \left( \sin^2 \frac{k\pi}{2M} + \sin^2 \frac{l\pi}{2M} \right) \psi^{(k,l)},
\]

where the operator \(-\Delta^{(h)} : U^{(h)} \rightarrow U^{(h)}\) is defined by formulæ (5.11), (5.10), and the space \(U^{(h)}\) contains all discrete functions specified on the interior subset of the grid (5.104), i.e., for \(m_1, m_2 = 1, 2, \ldots, M - 1\). Note that even though the functions \(\psi^{(k,l)}\) of (5.105) are formally defined on the entire grid (5.104) rather than on its interior subset only, we can still assume that \(\psi^{(k,l)} \in U^{(h)}\) with no loss of generality, because the boundary values of \(\psi^{(k,l)}\) are fixed: \(\psi_{m_1, m_2}^{(k,l)} = 0\) for \(m_1, m_2 = 0, M\).

Equalities (5.108) imply that the grid functions \(\psi^{(k,l)}\) that form an orthonormal basis in the space \(U^{(h)}\), see formulæ (5.106), are eigenfunctions of the operator \(-\Delta^{(h)} : U^{(h)} \rightarrow U^{(h)}\), whereas the numbers:

\[
\lambda_{kl} = \frac{4}{h^2} \left( \sin^2 \frac{k\pi}{2M} + \sin^2 \frac{l\pi}{2M} \right), \quad k, l = 1, 2, \ldots, M - 1,
\]

are the corresponding eigenvalues. As all the eigenvalues (5.109) are real (and positive), the operator \(-\Delta^{(h)}\) is self-adjoint, i.e., symmetric (and positive definite). Indeed, for an arbitrary pair \(v, w \in U^{(h)}\) we first use the expansion (5.107):

\[
v = \sum_{k,l=1}^{M-1} c_{kl} \psi^{(k,l)}, \quad w = \sum_{r,s=1}^{M-1} d_{rs} \psi^{(r,s)},
\]

and then substitute these expressions into formulæ (5.108), which yields [with the help of formulæ (5.106)]:

\[
(-\Delta^{(h)} v, w) = \left( -\Delta^{(h)} \left( \sum_{k,l=1}^{M-1} c_{kl} \psi^{(k,l)} \right) , \sum_{r,s=1}^{M-1} d_{rs} \psi^{(r,s)} \right)
= \left( \sum_{k,l=1}^{M-1} c_{kl} \lambda_{kl} \psi^{(k,l)} , \sum_{r,s=1}^{M-1} d_{rs} \psi^{(r,s)} \right)
= \sum_{k,l=1}^{M-1} c_{kl} \lambda_{kl} d_{kl}
= \left( \sum_{k,l=1}^{M-1} c_{kl} \psi^{(k,l)} , -\Delta^{(h)} \left( \sum_{r,s=1}^{M-1} d_{rs} \psi^{(r,s)} \right) \right)
= (v, -\Delta^{(h)} w).
\]

According to the general scheme of the Fourier method given by formulæ (5.94), (5.95), and (5.96), the solution of problem (5.97) can be written in the form:

\[
u_{m_1, m_2} = \sum_{k,l=1}^{M-1} \frac{f_{kl}}{\lambda_{kl}} 2 \sin \frac{k\pi m_1}{M} \sin \frac{l\pi m_2}{M}, \quad \text{(5.110)}
\]

\(^6\)See also the analysis on page 269.