where \( x \in \mathbb{L} \) is the vector of unknowns and \( f \in \mathbb{L} \) is a given right-hand side. Let \( \Delta f \in \mathbb{L} \) be a perturbation of the right-hand side that leads to the perturbation \( \Delta x \in \mathbb{L} \) of the solution so that:

\[
A(x + \Delta x) = f + \Delta f. \tag{5.35}
\]

Then the relative error of the solution \( \frac{\|\Delta x\|}{\|x\|} \) satisfies the inequality:

\[
\frac{\|\Delta x\|}{\|x\|} \leq \mu(A) \frac{\|\Delta f\|}{\|f\|}, \tag{5.36}
\]

where \( \mu(A) \) is the condition number of the operator \( A \), see formula (5.31). Moreover, there are particular \( f \in \mathbb{L} \) and \( \Delta f \in \mathbb{L} \) for which (5.36) transforms into a precise equality.

**PROOF**

Formulae (5.34) and (5.35) imply that \( A\Delta x = \Delta f \), and consequently, \( \Delta x = A^{-1} \Delta f \). Let us also employ the expression \( Ax = f \) that defines the original system itself. Then,

\[
\frac{\|\Delta x\|}{\|x\|} = \frac{\|A^{-1} \Delta f\|}{\|x\|} = \frac{\|Ax\| \|A^{-1} \Delta f\|}{\|x\| \|Ax\| \|A^{-1} \Delta f\|} = \frac{\|Ax\| \|A^{-1} \Delta f\|}{\|f\| \|\Delta f\|}. \tag{5.37}
\]

According to the definition of the operator norm, see formula (5.24), we have:

\[
\frac{\|Ax\|}{\|x\|} \leq \|A\| \quad \text{and} \quad \frac{\|A^{-1} \Delta f\|}{\|\Delta f\|} \leq \|A^{-1}\|. \tag{5.38}
\]

Combining formulae (5.37) and (5.38), we obtain for any \( f \in \mathbb{L} \) and \( \Delta f \in \mathbb{L} \):

\[
\frac{\|\Delta x\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|\Delta f\|}{\|f\|} = \mu(A) \frac{\|\Delta f\|}{\|f\|}, \tag{5.39}
\]

which means that inequality (5.36) holds.

Furthermore, if \( \Delta f \) is the specific vector from the space \( \mathbb{L} \) for which

\[
\frac{\|A^{-1} \Delta f\|}{\|\Delta f\|} = \|A^{-1}\|,
\]

and \( f = Ax \) is the element of \( \mathbb{L} \) for which

\[
\frac{\|Ax\|}{\|x\|} = \|A\|,
\]

then expression (5.37) coincides with inequalities (5.39) and (5.36) that transform into precise equalities for these particular \( f \) and \( \Delta f \).