5.1 Different Forms of Consistent Linear Systems

A linear equation with respect to the unknowns \( z_1, z_2, \ldots, z_n \) is an equation of the form

\[
\alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_n z_n = t,
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_n \), and \( t \) are some given numbers (constants).

5.1.1 Canonical Form of a Linear System

Let a system of \( n \) linear algebraic equations be specified with respect to as many unknowns. This system can obviously be written in the following canonical form:

\[
\begin{align*}
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= f_1, \\
\vdots &\quad \vdots \\
a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n &= f_n.
\end{align*}
\]

Both the unknowns \( x_j \) and the equations in system (5.1) are numbered by the consecutive integers: 1, 2, \ldots, \( n \). Accordingly, the coefficients \( a_{ij} \) in front of the unknowns have double subscripts: \( i, j = 1, 2, \ldots, n \), where \( i \) reflects on the number of the equation and \( j \) reflects on the number of the unknown. The right-hand side of equation number \( i \) is denoted by \( f_i \) for all \( i = 1, 2, \ldots, n \).

From linear algebra we know that system (5.1) is consistent, i.e., has a solution, for any choice of the right-hand sides \( f_i \) if and only if the corresponding homogeneous system\(^2\) only has a trivial solution: \( x_1 = x_2 = \ldots = x_n = 0 \). For the homogeneous system to have only the trivial solution, it is necessary and sufficient that the determinant of the system matrix \( A \):

\[
A = \begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \ldots & a_{nn}
\end{bmatrix}
\]

\[(5.2)\]

differ from zero: \( \det A \neq 0 \). The condition \( \det A \neq 0 \) that guarantees consistency of system (5.1) for any right-hand side also implies uniqueness of the solution.

Having introduced the matrix \( A \) of (5.2), we can rewrite system (5.1) as follows:

\[
\begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \ldots & a_{nn}
\end{bmatrix} \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
f_1 \\
\vdots \\
f_n
\end{bmatrix},
\]

\[(5.3)\]

or, using an alternative shorter matrix notation:

\[
Ax = f.
\]

\(^2\)Obtained by setting \( f_i = 0 \) for all \( i = 1, 2, \ldots, n \).