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Nilpotent Lie and Leibniz Algebras

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We extend results on finite dimensional nilpotent Lie algebras to Leibniz algebras and counterexamples to others are found. One generator algebras are used in these examples and are investigated further.

1. INTRODUCTION

Results on nilpotent Lie algebras have been extended to Leibniz algebras by various authors. We focus on necessary and sufficient conditions for determining whether a Leibniz algebra $A$ is nilpotent. Previous results on this topic include: Engel’s Theorem ([2], [6], [8], [13]), the nilpotency of algebras which admit a prime period automorphism without nonzero fixed points, and the equivalence of (a) nilpotency, (b) the normalizer condition, (c) the right normalizer condition, (d) that all maximal subalgebras are ideals, and (e) that all maximal subalgebras are right ideals (which follows from material in [5]).

In this work we show that a Leibniz algebra $A$ is nilpotent exactly when it has a nilpotent ideal $N$ such that $A/N^2$ is nilpotent, when $A$ satisfies condition $k$ (as defined in [11]), and when $A$ is an $S^*$ algebra (using a generalized version of the definition in [10]). We also show several nilpotency conditions which fail to extend from Lie to Leibniz algebras, using cyclic Leibniz algebras as counterexamples. We study these algebras in Section 4, characterizing maximal subalgebras, Frattini subalgebras, Cartan subalgebras and various ideals of these algebras in terms of annihilating polynomials for left multiplication by the generator. Finally, we consider Leibniz algebras whose center or right center is one dimensional and extend non-embedding results from Lie theory ([9], [14]). There are results in [1] for when the left center is one dimensional. In this work we consider only finite dimensional Leibniz algebras.
2. PRELIMINARIES

Let \( \Phi(A) \) be the Frattini subalgebra of the Leibniz algebra \( A \), which is the intersection of all maximal subalgebras of \( A \). As in Lie theory, \( \Phi(A) \) is an ideal when the algebra is of characteristic 0 [7], but not generally, even if the algebra is solvable [7], which is counter to the case for solvable Lie algebras.

We will consider left Leibniz algebras, following Barnes [5]. Hence a Leibniz algebra is an algebra that satisfies the identity \( x(yz) = (xy)z + y(xz) \). We consider only finite dimensional algebras over a field. Let \( A \) be a Leibniz algebra. The center of \( A \) will be denoted by \( Z(A) \) and \( R(A) \) will be the right center, \( \{ a \in A : Aa = 0 \} \). We let \( A^2 = AA \) and define the lower central series by \( A^{j+1} = AA^j \). It is known that \( A^j \) is the space of all linear combination of products of \( j \) elements no matter how associated. Thus it is often sufficient to consider only left-normed products of elements. \( A \) is nilpotent of class \( t \) if \( A^{t+1} = 0 \) but \( A^t \neq 0 \). When \( A \) has class \( t \), then \( A^t \subset Z(A) \).

3. EXTENDING LIE NILPOTENCY PROPERTIES

In this section we extend several properties equivalent to nilpotency in Lie theory to Leibniz algebras. We show two properties that hold for Leibniz algebras as in the Lie algebra case, one property which fails to extend, and one which holds for Leibniz algebras when we generalize a definition from Lie theory.

Theorem 3.1. Let \( A \) be a Leibniz algebra and \( N \) be a nilpotent ideal of \( A \). Then \( A \) is nilpotent if and only if \( A/N^2 \) is nilpotent.

The proof of this theorem is identical to the Lie algebra case, shown in [10], and we omit it here. The following corollary is a consequence of the proof of Theorem 3.1.

Corollary 3.2. Let \( A \) be a Leibniz algebra and \( N \) be an ideal of \( A \). Suppose that \( N \) is nilpotent of class \( c \) and \( A/N^2 \) is nilpotent of class \( d + 1 \). Then \( A \) is nilpotent of class at most \( \binom{c+1}{2} d - \left( \frac{d}{2} \right) \).

We say that a Leibniz algebra \( A \) satisfies condition \( k \) if the only subalgebra \( K \) of \( A \) with the property \( K + A^2 \) is \( K = A \).

Theorem 3.3. Let \( A \) be a Leibniz algebra. Then \( A \) is nilpotent if and only if \( A \) satisfies condition \( k \).

The proof of this result is the same as in the Lie algebra case [11], and follows from the result of [5] mentioned in the introduction.

The following two examples are of concepts that when applied to Leibniz algebras are no longer equivalent to nilpotency. Let \( S \) be a subset of the Lie algebra \( A \). The normal closure, \( S^A \), of \( S \) is the smallest ideal of \( A \) that contains \( S \). A Lie algebra, \( A \), is nilpotent if and only if there is exactly one non-zero nilpotent subalgebra whose normal closure is \( A \) [11] (there is a requirement that the dimension of \( A \) is large compared to the cardinality of the field). This result fails for Leibniz algebras when the normal closure is defined as above.
Example 3.4. Let \( A \) be a Leibniz algebra with basis \( \{ a, a^2 \} \), and \( aa^2 = a^2 \). \( H = \text{spn}\{a - a^2\} \) is a nilpotent subalgebra and, since \( Ha \) is not contained in \( H \), \( H^A = A \). \( H \) is unique with respect to this property. For any proper subalgebra is of the form \( J = \text{spn}\{a + xa^2\} \), and \( J \) is a subalgebra if and only if \( x = -1 \). Therefore, although \( A \) is not nilpotent, the nilpotent subalgebra whose normal closure is \( A \) is unique.

A Lie algebra is an \( S^* \) algebra if each non-abelian subalgebra \( H \) has \( \dim(H/H^2) \geq 2 \). A Lie algebra is an \( S^* \) algebra if and only if it is nilpotent \[10\]. This result does not extend directly to Leibniz algebras.

Example 3.5. Let \( A \) be a Leibniz algebra with basis \( \{ a, a^2 \} \), and \( aa^2 = 0 \). Then \( A \) is nilpotent, however \( \dim(A/A^2) = 1 \).

We can alter the definition of an \( S^* \) algebra to obtain a property equivalent to nilpotency which restricts to the original definition in the Lie algebra case. Define a Leibniz algebra to be an \( S^* \) algebra if every non-abelian subalgebra \( H \) has either \( \dim(H/H^2) \geq 2 \) or \( H \) is nilpotent and generated by one element.

Theorem 3.6. A Leibniz algebra is an \( S^* \) algebra if and only if it is nilpotent.

Lemma 3.7. Let \( A \) be a non-abelian nilpotent Leibniz algebra. Then either \( \dim(A/A^2) \geq 2 \) or \( A \) is generated by one element.

Proof. Since \( A \) is nilpotent, \( A^3 = \Phi(A) \), the Frattini subalgebra of \( A \). Clearly, \( \dim(A/A^2) \neq 0 \) since \( A \) is nilpotent. If \( \dim(A/A^2) = 1 \), then \( A \) is generated by one element. Otherwise, \( \dim(A/A^2) \geq 2 \). \( \square \)

Lemma 3.8. If \( A \) is not nilpotent but all proper subalgebras of \( A \) are nilpotent, then \( \dim(A/A^2) \leq 1 \).

Proof. Suppose that \( \dim(A/A^2) \geq 2 \). Then there exist distinct maximal subalgebras, \( M \) and \( N \) which contain \( A^2 \). Hence \( M \) and \( N \) are ideals and \( A = M + N \) is nilpotent, a contradiction. \( \square \)

Proof of Theorem 3.6. If \( A \) is nilpotent, then every subalgebra is nilpotent, so \( A \) is an \( S^* \) algebra by Lemma 3.7. Conversely, suppose that there exists an \( S^* \) algebra that is not nilpotent. Let \( A \) be one of smallest dimension. All proper subalgebras of \( A \) are \( S^* \) algebras, hence are nilpotent. Thus \( \dim(A/A^2) \leq 1 \) by Lemma 3.8. Since \( A \) is an \( S^* \) algebra, it is generated by one element and is nilpotent, a contradiction. \( \square \)

4. CYCLIC LEIBNIZ ALGEBRA

In the last section, we found that Leibniz algebras generated by one element provide counterexamples to the extension of several results from Lie to Leibniz algebras. It would seem to be of interest to find properties of these algebras. In this section, we study them in their own right.

Let \( A \) be a cyclic Leibniz algebra generated by \( a \), and let \( L_a \) denote left multiplication on \( A \) by \( a \). Let \( \{ a, a^2, \ldots, a^g \} \) be a basis for \( A \) and \( aa^n = a_1 a + \cdots + a_g a^{g+1} \).
Corollary 4.2. \( \Phi(A) = 0 \) if and only if \( p(x) \) is the product of distinct prime factors.
Corollary 4.3. The maximal subalgebras of $A$ are precisely the null spaces of $r_j(L_a)$, where $r_j(x) = p(x)/p_j(x)$ for $j = 1, \ldots, s$.

Now let $A_0$ and $A_1$ be the Fitting null and one components of $L_a$ acting on $A$. Since $L_a$ is a derivation of $A$, $A_0$ is a subalgebra of $A$. $L_a$ acts nilpotently on $A_0$ and $L_b = 0$ when $b \in A^2$. Therefore, for each $c \in A$, $L_c$ is nilpotent on $A_0$, and $A_0$ is nilpotent by Engel's theorem. Let $a = b + c$, where $b \in A_0$ and $c \in A$. Then $L_a = L_b$ since $A_1 \subset A^2$ yields that $L_c = 0$. Then $bA_1 = aA_1 = A_1$. For any nonzero $x \in A_1$, $bx$ is nonzero in $A_1$, and $x$ is not in the normalizer of $A_0$. Hence $A_1 \cap N_x(A_0) = 0$ and $A_0 = N_x(A_0)$. Hence $A_0$ is a Cartan subalgebra of $A$.

Conversely, let $C$ be a Cartan subalgebra of $A$ and $c \in C$. Then $c = d + e$, $d \in A_0$, and $e \in A_1$. Since $A_1 \subset A^2$, $A_1A_1 = A_1A_0 = 0$, $A_0A_1 = A_1$, and $A_0A_0 \subset A_0$. Therefore, $A_1$ is an abelian subalgebra of $A$. Now $0 = \mathcal{L}_1^{n}(c) = \mathcal{L}_0^{n}(c) = \mathcal{L}_0^{n}(d + e) = \mathcal{L}_0^{n}(e) = \mathcal{L}_0^{n}(c)$, where we used that $eA = 0$ and $d \in A_0$ which is nilpotent. Since $L_c$ is non-singular on $A_1$, $e = 0$ and $c = d$. Hence $C \subset A_0$. Since $C$ is a Cartan subalgebra and $A_0$ is nilpotent, $C = A_0$, and $A_0$ is the unique Cartan subalgebra of $A$.

Theorem 4.4. $A$ has a unique Cartan subalgebra. It is the Fitting null component of $L_a$ acting on $A$.

Using these same ideas, the following corollary can be shown.

Corollary 4.5. The minimal ideals of $A$ are precisely $I_j = \{ b \in A : p_j(L_a)(b) = 0 \}$ for $j > 1$ and, if $n_1 > 1$, $I_1 = \{ b \in A : p_1(L_a)(b) = 0 \}$.

Corollary 4.6. $\text{Assoc}(A) = \{ b \in A : u(L_a)(b) = 0 \}$, where $u(x) = p_2(x) \ldots p_j(x)$ if $n_1 = 1$ and $u(x) = p_1(x) \ldots p_j(x)$ otherwise.

Corollary 4.7. The unique maximal ideal of $A$ is $M_1 = \{ b \in A : t(L_a)(b) = 0 \}$, where $t(x) = p(x)/p_1(x)$.

5. NON-EMBEDDING

Non-abelian Lie algebras with one dimensional centers cannot be embedded as certain ideals in the derived algebra of any nilpotent Lie algebra [9], [14]. In this section, we extend these results to Leibniz algebras.

Let $A$ be a Leibniz algebra. Define the upper central series as usual; that is, let $Z_1(A) = \{ z \in A : zA = Az = 0 \}$ and inductively, $Z_{i+1}(A) = \{ z \in A : Az \subset Z_i(A) \}$. If $A$ is an ideal in a Leibniz algebra $N$, then the terms in the upper central series of $A$ are ideals in $N$. Suppose that $A$ is nilpotent of dimension greater than one and $\dim(Z_1(A)) = 1$. We will show that $A$ cannot be any $N'$, $i \geq 2$, for any nilpotent Leibniz algebra $N$. This is an extension of the Lie algebra result in [9]. Suppose to the contrary that $A = N'$, where $N$ is nilpotent of class $t$, and let $z$ be a basis for $Z_1(A)$. For $n \in N$, $nz = z_n z$ and $zn = z_{n,z}$. If one of these coefficients is not 0, then $N$ is not nilpotent. Hence, $Z_1(A) \subset Z_1(N)$. Since $N'$ is an ideal in $A$, $N' \subset Z_1(A)$. Then, since $\dim(Z_1(A)) = 1, N' = Z_1(A)$. Our initial assumptions guarantee that $N'' \subset A$. Hence there exists a $y \in N'' \subset A, y \not \in Z_1(A)$, such that $yu = z_{y,z} z$ and $uy = z_{y,z} z$ for all $u \in N$. Let $w$ also be in $N$. Then $y(uw) = (yu)w + u(yw) = z_{w,z} w +
Let $A$ be a Leibniz algebra. Define $R_i(A) = \{ r \in A : Ar = 0 \}$ and, inductively, $R_{j+1}(A) = \{ r \in A : Ar \subset R_j(A) \}$. The $R_i(A)$ are left ideals of $A$. Let $B$ be an ideal in $A$, and let $L$ be the homomorphism from $A$ into the Lie algebra of derivations of $B$ given by $L(a) = L_a$, left multiplication of $B$ by $a$. Let $E(B, A)$ be the image of $L$. $E(B, A)$ is a Lie algebra, and $E(B, B)$ is an ideal in $E(B, A)$. For any left ideal, $C$, of $A$ that is contained in $B$, let $E(B, A, C) = \{ E \in E(B, A) : E(C) = 0 \}$. $E(B, A, C)$ is an ideal in $E(B, A)$, and $E(B, A)/E(B, A, C)$ is isomorphic to $E(C, A)$.

**Theorem 5.2.** Let $B$ be a nilpotent Leibniz algebra with $\dim(R_1(B)) = 1$ and $\dim(B) \geq 2$. Then $B$ is not an ideal of any Leibniz algebra $A$ in which $B \subset \Phi(A)$.

**Proof.** Suppose that $B$ is an ideal in $A$ that contradicts the theorem. Then $E(B, B) = L(B) \subset L(\Phi(A)) \subset \Phi(L(A)) = \Phi(E(B, A))$. Let $\{ z_1, z_2, \ldots, z_k \}$ be a basis for $R_2(B)$ and $\{ z_k \}$ be a basis for $R_1(B)$. Let $\Pi$ be the restriction map from $E(B, A)$ to $E(R_2(B), A)$. Since $(E(B, B) + E(B, A, R_1(B))) / E(B, A, R_1(B)) = E(B, B) / (E(B, A, R_1(B)) \cap E(B, B)) = E(B, B) / E(B, B, R_1(B)) = E(R_2(B), B)$, it follows that $E(R_2(B), B) = \Pi(E(B, B)) \subset \Pi(\Phi(E(B, A))) \subset \Phi(E(R_2(B), A))$.

Now we show that $E(R_2(B), B)$ is not contained in $\Phi(E(R_2(B), A))$ by showing that $E(R_2(B), B)$ is complemented by a subalgebra in $E(R_2(B), A)$. For $i = 1, \ldots, k$, let $e_i(z) = \delta_{ij}z_j$ for $j = i, \ldots, k$, where $\delta_{ij}$ is the Kronecker delta. Let $S = \text{spn}\{ e_1, \ldots, e_k \}$. We claim that $S = E(R_2(B), B)$. Since $BR_2(B) \subset R_1(B)$, it follows that $E(R_2(B), B) \subset S$. To show equality, we show $\dim(E(R_2(B), B)) = k - 1 = \dim(S)$. For $x \in B$, $L_x$ induces a linear functional on $R_2(B)$, that is, $E(R_2(B), B)$ is contained in the dual of $R_2(B)$. Hence $\dim(E(R_2(B), B) = \dim(R_2(B)) - \dim(R_2(B)^0)$, where $R_2(B)^0 = \{ z \in R_2(B) : L_x(z) = 0 \}$ for all $x \in B = R_1(B)$. Hence $\dim(E(R_2(B), B)) = k - 1 = \dim(S)$, and $S = E(R_2(B), B)$.

We now show that $S$ is complemented in $E(R_2(B), A)$. Let $M = \{ E \in E(R_2(B), A) : E(z_i) = \sum_{j=1}^{i-1} \lambda_{ij}z_j, \lambda_{ij} \in F, i = 1, \ldots, k - 1 \}$. $M$ is a subalgebra of $E(R_2(B), A)$ and $M \cap S = 0$. We claim that $M + S = E(R_2(B), A)$. Let $E \in E(R_2(B), A)$.

Now $E = (E - \sum_{i=1}^{k-1} \lambda_{ik}e_k) + (\sum_{i=1}^{k-1} \lambda_{ik}e_k) \in M + S$. Therefore, $E(R_2(B), A) = M + S$. Suppose that $M = 0$. Then $E(R_2(B), A) = E(R_2(B), B)$, which contradicts $E(R_2(B), B) \subset \Phi(E(R_2(B), A))$. Thus, $M \neq 0$. Hence, $S$ is complemented in $E(R_2(B), A)$, which contradicts $S \subset \Phi(E(R_2(B), A))$. This contradiction establishes the result.

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\[ u\alpha_u z = 0. \] Similarly \((uu)y = 0\). Since \(A \subset N^i\), it follows that \(y\) is in the center of \(A\). Since \(y\) and \(z\) are linearly independent, this is a contradiction. Hence, we have the following theorem.

**Theorem 5.1.** Let \(A\) be a nilpotent non-abelian Leibniz algebra with one-dimensional center. Then \(A\) cannot be any \(N^i, i \geq 2, \) for any nilpotent Leibniz algebra \(N\).
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REFERENCES